Theta representations on covering groups

Author: Yuanqing Cai

Boston College Electronic Thesis or Dissertation, 2017

Theta representations on covering groups

Yuanqing Cai

A dissertation

submitted to the Faculty of

the Department of Mathematics

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Boston College
Morrissey College of Arts and Sciences
Graduate School

August 2017

Theta representations on covering groups

Yuanqing Cai

Thesis advisor: Sololom Friedberg

Kazhdan and Patterson constructed generalized theta representations on covers of general linear groups as multi-residues of the Borel Eisenstein series. For the double covers, these representations and their (degenerate-type) unique models were used by Bump and Ginzburg in the Rankin-Selberg constructions of the symmetric square L-functions for $GL(r)$. In this thesis, we study two other types of models that the theta representations may support. We first discuss semi-Whittaker models, which generalize the models used in the work of Bump and Ginzburg. Secondly, we determine the unipotent orbits attached to theta functions, in the sense of Ginzburg. We also determine the covers for which these models are unique. We also describe briefly some applications of these unique models in Rankin-Selberg integrals for covering groups.

Contents

Citations to Previous Work

Most of this thesis (Chapter 2-5) is based on [Cai16b]. Chapter 6 is based on [CFGK16] and [CFGK17].

Acknowledgments

I would first and foremost like to thank my advisor Solomon Friedberg for his guidance, support, and patience throughout my time at Boston College. Without his guidance, I could not have got through this Ph.D. program so smoothly. I also enjoyed (and will still enjoy) stimulating conversations with David Ginzburg and Eyal Kaplan; it's a pleasure to acknowledge their help and influence on this work.

My time at BC has been quite enjoyable. Many people have contributed, sometimes in an unexpected way. My mathematical growth benefited a lot from conversations with a number of talented people, especially Ben Howard and Mark Reeder. I wish to thank them for teaching me lots of important stuff.

I am deeply grateful to the students, postdocs, and professors of the Mathematics Department for creating such an amazing academic community.

I wish to thank my wife Chang Liu, for her love, encouragement, care, and support over all these years.

To my parents

Sebiao Cai

and

Shaohui Chen

Chapter 1

Introduction

Since the pioneering work of Shimura on half-integral weight modular forms, non-linear central extensions of algebraic groups have played a substantial role in number theory and representation theory. The most well-known examples are the theory of theta liftings and dual pairs, which lie on the metaplectic double cover of the symplectic group.

To construct higher theta functions, we need to look at its beautiful connection with Eisenstein series. One can construct Borel Eisenstein series on the double cover of $SL(2)$ without any difficulty. As in the non-metaplectic case, this Eisenstein series possesses meromorphic continuation and has a right-most pole. If the residue is taken, one obtains, miraculously, a constant multiple of the Jacobi theta function. This approach was used by Kubota to begin the investigation of generalized theta functions and Eisenstein series. It was continued by Patterson (who used the cubic cover of $SL(2)$ to study Kummer's conjecture on the distribution of cubic Gauss sums), by Kazhdan and Patterson and many others.

The general subject of metaplectic forms, i.e. automorphic functions on the metaplectic group, and the specific examples of higher theta functions and Eisenstein series, has proved to be highly interesting for its own sake. It also has many applications towards number theory and the theory of non-metaplectic automorphic forms. This is not a surprising fact. After all, the nth power reciprocity law, a deep theorem in arithmetic, underlies the existence of the metaplectic groups.

In this thesis, we focus the representation-theoretic aspects of the theta representations. In particular, we are interested in models the theta representations might support. It is a widely-known and much used fact that the vast majority of irreducible admissible representations of $GL(r)$ have a unique Whittaker models. This is false in general and it is the main obstruction one has to deal with. It is an interesting question to locate the rare "exceptional" representations that do support a unique Whittaker model. For Kazhdan-Patterson coverings, this is done in [KP84].

Let F be a number field containing a full set of nth roots of unity. Let $\mathbb{A}$ be its adele ring. Let $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ be a metaplectic n-fold cover of the general linear group. Kazhdan-Patterson [KP84] constructed generalized theta representations Θ_r on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ as multi-residues of the Borel Eisenstein series. The local theta representations were also constructed, as the Langlands quotient of reducible principal series representations. They showed that (both globally and locally) the generalized theta representations are generic if and only if $n \geq r$; and uniqueness of Whittaker models holds if and only if $n = r$ or $r + 1$ (when $n = r + 1$, the uniqueness property only holds for certain covers). The theta representations and their unique models have been used to construct Rankin-Selberg integrals for symmetric powers L-functions for the general linear groups; see Shimura [Shi75], Gelbart-Jacquet [GJ78], Patterson-Piatetski-Shapiro [PPS89], Bump-Ginzburg [BG92], Bump-Ginzburg-Hoffstein [BGH96], and Takeda [Tak14].

Less is known for theta representations on higher covers and their applications. In this thesis, we are interested in the theta representations when they are non-generic. Our goal is to find other models that the theta representations may support. We also look for special cases when these models are unique – this is a rare phenomenon, as we indicate above. In the last chapter of this thesis, we give an application of these unique models in Rankin-Selberg integrals for covering groups. We describe an example of the doubling constructions for covering groups – this is joint work with Friedberg, Ginzburg and Kaplan. This construction also sheds light on how to develop other Rankin-Selberg integrals for covering groups.

1.1 Statements of Main Results

We first introduce a generalization of the Whittaker coefficients, which we call semi-Whittaker coefficients. Let $\lambda = (r_1 \cdots r_k)$ be a partition of r. Let P_λ be the standard parabolic subgroup of GL_r whose Levi subgroup $M \cong \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$. Let

U_λ be its unipotent radical. Fix a nontrivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. Let $\psi_\lambda : U(F)\backslash U(\mathbb{A}) \to \mathbb{C}^\times$ be the character such that it acts as ψ on the simple positive root subgroups contained in M, and acts trivially otherwise. The λ-semi-Whittaker coefficient of $\theta \in \Theta_r$ is defined to be

$$\int\limits_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\, du.$$

When the partition is $\lambda = (r)$, this recovers the usual Whittaker coefficients.

Theorem 1.1.1.

(1) *If there is an $r_i > n$, then*

$$\int\limits_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\, du$$

is zero for all choices of data.

(2) *If $r_i \leq n$ for all i, then*

$$\int\limits_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\, du$$

is nonzero for some choice of data.

(3) *When $r = mn$, i.e. when the rank is a multiple of the degree, and the partition is $\lambda = (n^m)$, then global uniqueness of λ-semi-Whittaker models holds.*

We remark that the local version of the above theorem is also established (see Corollary 3.6.5, 3.6.7, and Theorem 3.6.10). Indeed, parts (1) and (3) are proved by using the local results, and part (2) is proved by using a global argument. We also remark that when $n = 2$ and $\lambda = (2^k)$ or $(2^k 1)$ (depending on the parity of r), such coefficients and their uniqueness properties were already used in Bump and Ginzburg [BG92] in their work on symmetric square L-functions for $\mathrm{GL}(r)$.

The second type of Fourier coefficients we consider are the Fourier coefficients associated with unipotent orbits. The unipotent orbits of GL_r are parameterized by the partitions of r via the Jordan decomposition. Given a unipotent orbit $\mathcal{O}$, we can associate a set of Fourier coefficients; see Section 5 below. Roughly speaking, starting with a unipotent orbit $\mathcal{O}$, we can define a unipotent subgroup $U_2(\mathcal{O})$. Let

$\psi_{U_2(\mathcal{O})} : U_2(\mathcal{O})(F)\backslash U_2(\mathcal{O})(\mathbb{A}) \to \mathbb{C}^\times$ be a character which is in general position. The Fourier coefficient of $\theta \in \Theta_r$ we want to consider is

$$\int\limits_{U_2(\mathcal{O})(F)\backslash U_2(\mathcal{O})(\mathbb{A})} \theta(ug)\psi_{U_2(\mathcal{O})}(u)\ du.$$

When the unipotent orbit is $\mathcal{O} = (r)$, this also recovers the usual Whittaker coefficients. There is a partial ordering on the set of unipotent orbits. Our goal is to find the maximal unipotent orbit $\mathcal{O}(\Theta_r)$ that supports nonzero Fourier coefficients of Θ_r (see Definition 5.0.2 below). The main results for the Fourier coefficients associated with unipotent orbits are summarized as follows (Theorem 5.1.2, 5.2.1, and 5.1.11).

Theorem 1.1.2. (1) *Write $r = an + b$ such that $a \in \mathbb{Z}_{\geq 0}$ and $0 \leq b < n$. Then both locally and globally $\mathcal{O}(\Theta_r) = (n^a b)$.*

(2) *Let v be a finite place such that $|n|_v = 1$ and $\Theta_{r,v}$ is unramified. If $r = mn$ and $\mathcal{O} = (n^m)$, then*

$$\dim \mathrm{Hom}_{U_2(\mathcal{O})(F_v)}(\Theta_{r,v}, \psi_{U_2(\mathcal{O}),v}) = 1.$$

This unique model is valuable and it already finds applications in Rankin-Selberg integrals for covering groups. In the research announcement by Friedberg, Ginzburg, Kaplan and the author [CFGK16], the notion of Whittaker-Speh-Shalika representation was introduced (see Definition 6.1.1). Such representations are irreducible automorphic representations on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ and they possess unique functionals. The Whittaker-Speh-Shalika representations and their unique models are used in the generalization of the doubling methods to covering groups. The theta representations are examples of such representations.

Theorem 1.1.3 (Theorem 6.1.2)**.** *When $r = mn$, Θ_r is a Whittaker-Speh-Shalika representation of type (n, m).*

This unique functional also plays a role in a new-way integral (Euler products with non-unique models) for covering groups; see Ginzburg [Gin16].

1.2 Ideas of the Proofs

We now describe the ideas of the proofs. The proof of Theorem 1.1.1 is based on an induction in stages statement. We describe it in the global setup. Such an argument was also used in Bump-Friedberg-Ginzburg [BFG03] where they studied the Fourier coefficients of theta representations on the double covers of odd orthogonal groups. First of all, we can rewrite the λ-semi-Whittaker coefficients as

$$\int_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\, du = \int_{U\cap M(F)\backslash U\cap M(\mathbb{A})} \int_{U_\lambda(F)\backslash U_\lambda(\mathbb{A})} \theta(vug)\, dv\ \psi_\lambda(u)\, du.$$

The inner integral is actually a constant term of the theta function. To compute it, we compute the constant term of the Eisenstein series and use the fact that the multi-residue operator and the constant term operator commute. By the standard unfolding argument, the constant term of the Eisenstein series is a sum of Eisenstein series on $\widetilde{M}(\mathbb{A})$. After applying the multi-residue operator, only one term survives. This implies that the constant term of a theta function is actually a "theta function" on $\widetilde{M}(\mathbb{A})$.

Now we are facing a difficulty which did not appear in [BFG03]. In the double cover of the odd orthogonal case, the constant terms of theta functions give rise to a representation on the cover of the Levi subgroup. In that case, different blocks commute in $\widetilde{M}(\mathbb{A})$. Thus, one can take theta representations on each block and form the tensor product. It is shown that the tensor product of theta representations on each block is the same as the theta representation on $\widetilde{M}(\mathbb{A})$. We would like to seek an analogous result for the general linear group. However, in the general linear case, when we restrict the metaplectic cover to $\widetilde{M}$, the blocks never commute (except when $n = 2$). In fact, there is even no natural map between $\widetilde{M}$ and $\widetilde{\mathrm{GL}}_{r_1} \times \cdots \times \widetilde{\mathrm{GL}}_{r_k}$. This means that, starting with representations on the $\widetilde{\mathrm{GL}}_{r_i}$, there is no direct way to construct a representation of $\widetilde{M}$.

To overcome this difficulty, a construction called the metaplectic tensor product has been introduced (see Section 3.4 and 4.4). The local version is developed in Mezo [Mez04] and the global version is given in Takeda [Tak16, Takar]. Roughly speaking, the construction goes as follows (both locally and globally). Let $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$ be the subgroup of $\widetilde{\mathrm{GL}}_{r_i}$, consisting of those elements whose determinants are nth powers. Let $\widetilde{M}^{(n)}$ be the subgroup of $\widetilde{M}$ consisting of those elements such that the determinants of all the blocks are nth powers. The $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$'s commute in $\widetilde{M}$, and $\widetilde{M}^{(n)}$ is the direct product of $\widetilde{\mathrm{GL}}_{r_1}^{(n)}, \cdots, \widetilde{\mathrm{GL}}_{r_k}^{(n)}$ with amalgamated μ_n.

Now start with representations π_i on $\widetilde{\mathrm{GL}}_{r_i}$. We first restrict π_i to $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$, and pick an irreducible constituent $\pi_i^{(n)}$. Then we take the tensor product $\pi_1^{(n)}\otimes\cdots\otimes\pi_k^{(n)}$. This is a representation of $\widetilde{M}^{(n)}$. We then use induction to obtain a representation of $\widetilde{M}$. Extra care must be taken in order to establish the well-definedness and irreducibility of such constructions.

Theorem 1.2.1 (Rough form)**.** *Both locally and globally,*

$$\Theta_{\widetilde{M}} \cong \Theta_{r_1} \tilde{\otimes} \cdots \tilde{\otimes} \Theta_{r_k}.$$

The local version is given in Theorem 3.5.2, and the global version is Theorem 4.5.1. Once we have the induction in stages statement, Theorem 1.1.1 can be established by carefully analyzing the restriction and induction process. In the local setup, we give an explicit formula for the dimension of the twisted Jacquet module $J_{U,\psi_\lambda}(\Theta_r)$.

Theorem 1.1.2 is proved in Sections 5.1 and 5.2. The proof consists of two parts. The first part is to show that any unipotent orbit greater than or not comparable to $(n^a b)$ does not support any Fourier coefficients. The second part is to show that $(n^a b)$ actually supports a nonzero Fourier coefficient. The idea is to build a relation between the semi-Whittaker coefficients and the Fourier coefficients associated with unipotent orbits. Once we know enough information about the semi-Whittaker coefficients, the unipotent orbit attached to the representation can be determined.

Two tools play crucial roles in the proof. The first one is called root exchange. This allows us to replace the domain of integration with a slightly different one. The second one is the Fourier expansion. This allows us to enlarge the domain of integration if we know certain coefficients vanish (this is usually related to the vanishing of semi-Whittaker coefficients). When we combine these tools in a systematic way, vanishing and nonvanishing of Fourier coefficients associated with unipotent orbits can be related to the results on the semi-Whittaker coefficients. Furthermore, when n and b have the same parity, we actually establish an identity between these coefficients. In particular, Theorem 1.1.2 part (2) follows from Theorem 1.1.1 part (3).

1.3 Structure of this thesis

The remainder of this thesis is organized as follows. Section 2.1 introduces notations and defines metaplectic covers of general linear groups. Certain issues such as centers and maximal abelian subgroups are also discussed. The local theory of semi-Whittaker functionals is developed in Chapter 3. We first review the principal series representations and theta representations of $\widetilde{\mathrm{GL}}_r(F_v)$. In Section 3.2, we give an explicit description of the restriction of these representations to $\widetilde{\mathrm{GL}}_r^{(n)}(F_v)$. These results are used to provide examples of the metaplectic tensor product in Section 3.5. We then carefully analyze the construction and compute the dimensions of some twisted Jacquet modules in Section 3.6. Chapter 4 is devoted to the global theory. The nonvanishing part of Theorem 1.1.1 is proved in Theorem 4.6.1. In Chapter 5, we review the association of Fourier coefficients to a unipotent orbit. The unipotent orbit attached to the theta representations is determined in Sections 5.1 and 5.2. Section 5.1 introduces the local argument. The relation between semi-Whittaker coefficients and Fourier coefficients associated with unipotent orbit is established in a series of lemmas. Section 5.2 describes the corresponding global picture. In the last chapter, we describe the doubling construction for standard L-function for cubic cover of Sp_2.

Chapter 2

Notations and Preliminaries

2.1 Notations

Fix a positive integer n and let

$$\mu_n(F) = \{x \in F : x^n = 1\}$$

be the group of nth roots of unity in a field F. In this paper we always assume $|\mu_n(F)| = n$. Fix, once and for all, an embedding $\epsilon : \mu_n \to \mathbb{C}^\times$. We always write μ_n for $\mu_n(F)$, if there is no confusion. We often invoke the convention of omitting ϵ from the notation. All representations which we consider are representations where the central μ_n acts by scalars by the embedding ϵ . Such representations are called *genuine*.

Remark 2.1.1. It is safer for us to assume $|\mu_{2n}(F)| = 2n$ since our results rely on [KP84]. However, as we can see, all the arguments and results are still valid if we only assume $|\mu_n(F)| = n$.

If F is a non-Archimedean local field, we denote by $\mathfrak{o}$ the ring of integers of F. Let $|\cdot|_F$ be the normalized absolute value on F. Let

$$(\ ,\ ) = (\ ,\ )_{F,n} : F^\times \times F^\times \to \mu_n(F)$$

be the nth order Hilbert symbol. It is a bilinear form on $F^\times$ that defines a nondegenerate bilinear form on $F^\times/F^{\times n}$ and satisfies

$$(x, -x) = (x, y)(y, x) = 1, \qquad x, y \in F^\times.$$

When F is a number field, and v is a place of F, we denote by F_v the completion of F at v. When v is non-Archimedean, we let $\mathfrak{o}_v$ be the ring of integers of F_v.

For GL_r, let $B = TU$ be the standard Borel subgroup with unipotent radical U and maximal torus T. The set $\Phi = \{(i,j) : 1 \leq i \neq j \leq r\}$ is identified with the set of roots of GL_r in the usual way. Let Φ^+ denote the set of positive roots with respect to B.

For a partition $\lambda = (r_1 \cdots r_k)$ of r, let P_λ be the standard parabolic subgroup of GL_r whose Levi part M_λ is $\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$ embedded diagonally

$$(g_1, \cdots, g_k) \mapsto \mathrm{diag}(g_1, \cdots, g_k), \qquad g_i \in \mathrm{GL}_{r_i},$$

and let U_λ denote the unipotent radical of P_λ. We usually write M for M_λ when the partition is fixed. We usually write $m \in M$ by $m = \mathrm{diag}(g_1, \cdots, g_k)$ with $g_i \in \mathrm{GL}_{r_i}$. Let Φ_λ and Φ_λ^+ denote the set of roots and positive roots contained in M_λ, respectively.

Let W be set of all $r \times r$ permutation matrices. The Weyl group of GL_r is identified with W. We also identify W as the group of permutations of $\{1, 2, \cdots, r\}$ via

$$w = (\delta_{i,w(j)}).$$

The action of W on Φ is given by $w(i,j) = (w(i), w(j))$. For a Levi subgroup M_λ, let $W(M_\lambda)$ be the subset of permutation matrices contained in M_λ. The group $W(M_\lambda)$ is identified with the Weyl group of M_λ (as sets). Let

$$w^{M_\lambda} = \begin{pmatrix} & & & I_{r_1} \\ & & I_{r_2} & \\ & \iddots & & \\ I_{r_k} & & & \end{pmatrix} \in W.$$

The element w^{M_λ} sends $\mathrm{GL}_{r_k} \times \cdots \times \mathrm{GL}_{r_1}$ to M_λ.

For any group G and elements $g, h \in G$, we define ${}^g h = ghg^{-1}$. For a subgroup $H \subset G$ and a representation π of H, let ${}^g\pi$ be the representation of gHg^{-1} defined by ${}^g\pi(h') = \pi(g^{-1}h'g)$ for $h' \in gHg^{-1}$.

Let F be a local field. Let ψ be a nontrivial additive character on F. In this paper we need to consider several characters on various subgroups of U. We make the following convention. For a partition $(p_1 \cdots p_k)$ of $n' \leq n$, let $\Delta = \{i : 1 \leq i \leq n'\} \backslash \{p_1, p_1 + p_2, \cdots, p_1 + \cdots + p_k\}$. Let V be a subgroup of U such that V contains all the root subgroups associated to $\alpha = (i, i+1)$ for $i \in \Delta$. Let $\psi_{(p_1 \cdots p_k)} : V \to \mathbb{C}^\times$ be a character such that $\psi_{(p_1 \cdots p_k)}$ acts as ψ on the root subgroups associated to $\alpha = (i, i+1)$ for $i \in \Delta$, and acts trivially otherwise. Thus $\psi_{(r)}$ and $\psi_{(1^r)}$ are the usual Whittaker character and the trivial character on U, respectively. When F is a number field and ψ is a nontrivial additive character of $F\backslash\mathbb{A}$, these characters can be defined analogously.

2.2 The local metaplectic cover $\widetilde{\mathrm{GL}}_r(F)$

Let F be a local field of characteristic 0 that contains all the nth roots of unity. Associated to every 2-cocycle $\sigma : \mathrm{GL}_r(F) \times \mathrm{GL}_r(F) \to \mu_n(F)$, there is a central extension $\widetilde{\mathrm{GL}}_r(F)$ of $\mathrm{GL}_r(F)$ by μ_n satisfying an exact sequence

$$1 \to \mu_n \xrightarrow{\iota} \widetilde{\mathrm{GL}}_r(F) \xrightarrow{\mathbf{p}} \mathrm{GL}_r(F) \to 1.$$

We call $\widetilde{\mathrm{GL}}_r(F)$ a *metaplectic n-fold cover of* $\mathrm{GL}_r(F)$. As a set, we can realize $\widetilde{\mathrm{GL}}_r(F)$ as

$$\widetilde{\mathrm{GL}}_r(F) = \mathrm{GL}_r(F) \times \mu_n = \{(g, \zeta) : g \in \mathrm{GL}_r(F), \zeta \in \mu_n\}.$$

Notice that $\widetilde{\mathrm{GL}}_r(F)$ is not the F-rational points of an algebraic group, but this notation is standard. We use $\widetilde{\mathrm{GL}}_r$ to denote $\widetilde{\mathrm{GL}}_r(F)$. This abuse of notation is widely used in this paper, especially in the local setup. The embedding ι and the projection $\mathbf{p}$ are given by

$$\iota(\zeta) = (I_r, \zeta) \text{ and } \mathbf{p}(g, \zeta) = g$$

where I_r is the identity element of GL_r. The multiplication is defined in terms of σ as follows,

$$(g_1, \zeta_1) \cdot (g_2, \zeta_2) = (g_1 g_2, \zeta_1 \zeta_2 \sigma(g_1, g_2)).$$

For any subset $X \subset \mathrm{GL}_r$, let

$$\widetilde{X} = \mathbf{p}^{-1}(X) \subset \widetilde{\mathrm{GL}}_r.$$

We also fix the section $\mathbf{s} : \mathrm{GL}_r \to \widetilde{\mathrm{GL}}_r$ of $\mathbf{p}$ given by $\mathbf{s}(g) = (g, 1)$. Then for $g_1, g_2 \in \mathrm{GL}_r$,

$$\mathbf{s}(g_1)\mathbf{s}(g_2) = (g_1 g_2, \sigma(g_1, g_2)).$$

In [KP84], Kazhdan-Patterson provided 2-cocycles $\sigma^{(c)}$ parameterized by $c \in \mathbb{Z}/n\mathbb{Z}$ that exhaust all cohomology classes. They are related by

$$\sigma^{(c)}(g_1, g_2) = \sigma^{(0)}(g_1, g_2)(\det g_1, \det g_2)^c, \qquad g_1, g_2 \in \mathrm{GL}_r. \tag{2.1}$$

Here we want to take slightly different 2-cocycles. They are constructed in Banks-Levy-Sepanski [BLS99] and satisfy a block compatibility property. Let $\sigma^{(0)} = \sigma_r^{(0)}$, and $\sigma^{(c)} = \sigma_r^{(c)}$ be related to $\sigma_r^{(0)}$ by Eq. (2.1). Block compatibility means the following. If $r = r_1 + \cdots + r_k$ and $g_i, g'_i \in \mathrm{GL}_{r_i}$ for $i = 1, \cdots, k$, then

$$\begin{aligned} \sigma_r^{(c)}(\mathrm{diag}(g_1, \cdots, g_k), &\mathrm{diag}(g'_1, \cdots, g'_k)) \\ &= \left[\prod_{i=1}^{k} \sigma_{r_i}^{(c)}(g_i, g'_i)\right] \cdot \left[\prod_{i<j} (\det g_i, \det g'_j)^{c+1} (\det g_j, \det g'_i)^c\right]. \end{aligned}$$

Throughout the paper we fix the positive integers r and n and the modulus class $c \in \mathbb{Z}/n\mathbb{Z}$ and let $\sigma = \sigma_r^{(c)}$. Note that the restriction of σ to T is given by

$$\sigma(t, t') = \left[\prod_{i<j} (t_i, t'_j)\right] \cdot \prod_{i,j} (t_i, t'_j)^c$$

for $t = \mathrm{diag}(t_1, \cdots, t_r)$ and $t' = \mathrm{diag}(t'_1, \cdots, t'_r)$.

The group U splits in $\widetilde{\mathrm{GL}}_r$. In fact $\mathbf{s}\mid_U$ is an embedding of U in $\widetilde{\mathrm{GL}}_r$ ([McN12] Proposition 4.1). Let $K = \mathrm{GL}_r(\mathfrak{o})$. When $|n|_F = 1$, K also splits in $\widetilde{\mathrm{GL}}_r$ ([McN12] Theorem 4.2). There is a map $\kappa : K \to \mu_n$ such that $g \mapsto \kappa^*(g) = (g, \kappa(g))$ is a group homomorphism from K to $\widetilde{\mathrm{GL}}_r$. We denote its image by K^*. We shall fix κ such that κ^* is what Kazhdan-Patterson refer to as the canonical lift of K to $\widetilde{\mathrm{GL}}_r$. It is characterized by the property that

$$\mathbf{s}|_{T\cap K} = \kappa^*|_{T\cap K}, \mathbf{s}|_W = \kappa^*|_W, \text{ and } \mathbf{s}|_{U\cap K} = \kappa^*|_{U\cap K}.$$

([KP84] Proposition 0.1.3). The topology of $\widetilde{\mathrm{GL}}_r$ as a locally compact group is determined by this embedding.

2.3 Centers

The following lemma is Takeda [Tak16] Lemma 3.13, which is very useful for us.

Lemma 2.3.1. *Let F be a local field. Then for each $g \in \mathrm{GL}_r$ and $a \in F^\times$,*

$$\sigma_r(g, aI_r)\sigma_r(aI_r, g)^{-1} = (\det(g), a^{r-1+2cr}).$$

Lemma 2.3.2. *Let $n_1 = \gcd(n, 2rc + r - 1)$, and $n_2 = \dfrac{n}{n_1}$. Then the center of $\widetilde{\mathrm{GL}}_r$ is*

$$\begin{aligned} Z_{\widetilde{\mathrm{GL}}_r} =& \{(zI_r, \zeta) : z^{2rc+r-1} \in F^{\times n}\} \\ =& \{(zI_r, \zeta) : z \in F^{\times n_2}\}. \end{aligned}$$

The first part is proved in [KP84] Proposition 0.1.1, and the second part is proved in Chinta-Offen [CO13] Lemma 1.

The center of $\widetilde{T}$ is also determined in [KP84]. Let $\widetilde{T}^n = \{(t^n, \zeta) : t \in T\}$.

Lemma 2.3.3. *The center of $\widetilde{T}$ is $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}^n$.*

Let $\widetilde{\mathrm{GL}}_r^{(n)} := \{g \in \widetilde{\mathrm{GL}}_r : \det g \in F^{\times n}\}$. We are interested in group since it controls the representation theory of $\widetilde{\mathrm{GL}}_r$. Moreover, it plays a role in developing tensor products and parabolic inductions for metaplectic groups; see Section 3.4. Let $\widetilde{T}^{(n)} := \widetilde{\mathrm{GL}}_r^{(n)} \cap \widetilde{T}$. The centers of $\widetilde{\mathrm{GL}}_r^{(n)}$ and $\widetilde{T}^{(n)}$ behave better than the centers of $\widetilde{\mathrm{GL}}_r$ and $\widetilde{T}$.

Lemma 2.3.4. *The center of $\widetilde{\mathrm{GL}}_r^{(n)}$ is*

$$\begin{aligned} Z_{\widetilde{\mathrm{GL}}_r^{(n)}} = \widetilde{Z} \cap \widetilde{\mathrm{GL}}_r^{(n)} =& \{(aI_r, \zeta) : a^r \in F^{\times n}\} \\ =& \{(aI_r, \zeta) : a \in F^{\times \frac{n}{\gcd(n,r)}}\}. \end{aligned}$$

Proof. The first equality is immediate from Lemma 2.3.1. For the second equality, the proof is exactly the same as in [CO13] Lemma 1.

□

The proof of the following lemma is also straightforward.

Lemma 2.3.5. *The center of $\widetilde{T}^{(n)}$ is $Z_{\widetilde{\mathrm{GL}}_r^{(n)}}\widetilde{T}^n$.*

2.4 Maximal abelian groups

Maximal abelian subgroups of $\widetilde{T}$ play important role in the representation theory of $\widetilde{T}$. Let $\widetilde{T}_*$ be a maximal abelian subgroup $\widetilde{T}$. In Section 3.2-3.6, we assume that $\widetilde{T} \cap \widetilde{T}_*$ is a maximal abelian subgroup of $\widetilde{T}^{(n)}$, unless otherwise specified.

We now discuss the construction of maximal abelian subgroups. Given a maximal isotropic subgroup Ω of the Hilbert symbol, [KP84] Section 0.3 provides a way to construct maximal abelian subgroups of $\widetilde{T}$ under certain assumptions. When $|n|_F = 1$, $F^{\times n}\mathfrak{o}^\times$ is a maximal isotropic subgroup of the Hilbert symbol. Let

$$T_{\mathfrak{o}} = \{\mathrm{diag}(a_1, \cdots, a_r) \in T : v(a_i) \equiv 0 \mod n\}.$$

Then $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}_{\mathfrak{o}}$ is called the standard maximal abelian subgroup of $\widetilde{T}$, in the sense of [KP84] Section I.1. We use $\widetilde{T}_*^{\mathrm{st}}$ to denote this subgroup.

Remark 2.4.1. Notice that $\widetilde{T}_*^{\mathrm{st}} \cap \widetilde{T}^{(n)}$ is usually not a maximal abelian subgroup of $\widetilde{T}^{(n)}$, even for $n = 2$. When $n = 2, c = 0$, a "canonical" maximal abelian subgroup was introduced in Bump-Ginzburg [BG92]. The intersection of their maximal abelian subgroup and $\widetilde{T}^{(n)}$ is a maximal abelian subgroup of $\widetilde{T}^{(n)}$.

Let $\widetilde{T}_{\mathfrak{o}}^{(n)} = \widetilde{T}_{\mathfrak{o}} \cap \widetilde{\mathrm{GL}}_r^{(n)}$. The following proposition can be proved by imitating the argument in [KP84] Section 0.3.

Proposition 2.4.2. *The group $Z_{\widetilde{\mathrm{GL}}_r^{(n)}}\widetilde{T}_{\mathfrak{o}}^{(n)}$ is a maximal abelian subgroup of $\widetilde{T}^{(n)}$.*

Remark 2.4.3. Our calculation in Section 3.6 relies on the index $[\widetilde{T} : \widetilde{T}_*]$, which is an invariant of $\widetilde{T}$. This is computed by using the standard maximal abelian subgroup $\widetilde{T}_*^{\mathrm{st}}$.

Remark 2.4.4. When $|n|_F = 1$, we give an example of maximal abelian subgroup such that its intersection with $\widetilde{T}^{(n)}$ is $Z_{\widetilde{\mathrm{GL}}_r^{(n)}}\widetilde{T}_{\mathfrak{o}}^{(n)}$. Let

$$\widetilde{Z}_* = \{(zI_r, \zeta) \in \widetilde{Z} : z \in \mathfrak{o}^\times F^{\times \frac{n}{\gcd(n, r(2rc+r-1))}}\}$$

and

$$\widetilde{T}_{\mathfrak{o}}^{(n')} = \{a \in \widetilde{T}_{\mathfrak{o}} : \det(a) \in F^{\times \gcd(n,r)}\}.$$

Then $\widetilde{Z}_* Z_{\widetilde{\mathrm{GL}}_r^{(n)}}\widetilde{T}_{\mathfrak{o}}^{(n')} = \widetilde{Z}_*\widetilde{T}_{\mathfrak{o}}^{(n')}$ is a maximal abelian subgroup of $\widetilde{T}$ and its intersection with $\widetilde{T}^{(n)}$ is $Z_{\widetilde{\mathrm{GL}}_r^{(n)}}\widetilde{T}_{\mathfrak{o}}^{(n)}$.

Remark 2.4.5. When $|n|_F \neq 1$, it is usually difficult to construct maximal abelian subgroups of $\widetilde{T}$. However, when $n \mid r$, this situation is still nice in the following sense. Let Ω be an isotopic subgroup of the Hilbert symbol. Then by the construction in [KP84] Section 0.3,

$$\{(\mathrm{diag}(t_1, \cdots, t_r), \zeta) : t_i \in \Omega, \zeta \in \mu_n\}$$

is a maximal abelian subgroup of $\widetilde{T}$.

2.5 The global metaplectic cover $\widetilde{\mathrm{GL}}_r(\mathbb{A})$

Let F be a number field that contains all the nth roots of unity and $\mathbb{A}$ be the ring of adeles. To construct a metaplectic n-fold cover of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ of $\mathrm{GL}_r(\mathbb{A})$, we follow [Tak16] Section 2.2. The adelic 2-cocycle τ_r is defined by

$$\tau_r(g, g') = \prod_v \tau_{r,v}(g_v, g'_v),$$

for $g, g' \in \mathrm{GL}_r(\mathbb{A})$. Here, the local cocycle is obtained from the block-compatible cocycle, multiplied by a suitable coboundary. It can be shown that there is a section $\mathbf{s}_r : \mathrm{GL}_r(F) \to \widetilde{\mathrm{GL}}_r(\mathbb{A})$ such that $\mathrm{GL}_r(F)$ splits in $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. The center $Z_{\widetilde{\mathrm{GL}}_r(\mathbb{A})}$ of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ can be easily found by using the local results. As in the local case, we define

$$\widetilde{\mathrm{GL}}_r^{(n)}(\mathbb{A});= \{g \in \widetilde{\mathrm{GL}}_r(\mathbb{A}) : \det g \in \mathbb{A}^{\times n}\}.$$

2.6 Metaplectic cover of Levi subgroups

Let $\lambda = (r_1 \cdots r_k)$ be a partition of r. Let $M := M_\lambda$ be the Levi subgroup of GL_r described in Section 2.1. This section discusses metaplectic covers $\widetilde{M}$, both locally and globally. The 2-cocycle τ_r does not satisfy block-compatibility. To get round it, an equivalent cocycle τ_M was introduced in [Tak16] Section 3. We use this cocycle to define $\widetilde{M}$. Notice that the blocks GL_{r_i} don't commute with each other. Let $R = F$ if F is local and $R = \mathbb{A}$ is F is global. Define

$$\widetilde{M}^{(n)}(R) = \{(g_1 \cdots g_k, \zeta) : \det g_i \in R^{\times n}\}.$$

Let T be the maximal torus consisting of diagonal matrices. We write $T_i = T \cap \mathrm{GL}_{r_i}$, where GL_{r_i} is embedding in GL_r via

$$g \mapsto \mathrm{diag}(I_{r_1}, \cdots, g, \cdots, I_{r_k}).$$

The torus T_i can be viewed as a maximal torus of GL_{r_i}. Define $\widetilde{T}^{(n)} = \widetilde{T} \cap \widetilde{M}^{(n)}$. The following results are proved in [Tak16] Section 3. We omit the details.

Lemma 2.6.1. *The center of $\widetilde{M}(R)$ is*

$$Z_{\widetilde{M}(R)} = \left\{ \begin{pmatrix} a_1 I_{r_1} & & \\ & \ddots & \\ & & a_k I_{r_k} \end{pmatrix} : a_i^{r-1+2cr} \in R^{\times n} \textit{ and } a_1 \equiv \cdots \equiv a_k \mod R^{\times n}. \right\}$$

Remark 2.6.2. Notice that $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}^n = Z_{\widetilde{M}}\widetilde{T}^n$ and $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)} = Z_{\widetilde{M}}\widetilde{M}^{(n)}$.

Lemma 2.6.3. *The center of $\widetilde{M}^{(n)}$ is*

$$Z_{\widetilde{M}^{(n)}} = \left\{ \left(\begin{pmatrix} a_1 I_{r_1} & & \\ & \ddots & \\ & & a_k I_{r_k} \end{pmatrix}, \zeta \right) : a_i^{r_i} \in R^{\times n} \right\}.$$

Lemma 2.6.4. *The center of $\widetilde{T}^{(n)}$ is $Z_{\widetilde{M}^{(n)}}\widetilde{T}^n$.*

Next, assume F is local. We consider maximal abelian groups of $\widetilde{T}^{(n)}$. For $1 \leq i \leq k$, let $\widetilde{T}_{*,i}^{(n)}$ be a maximal abelian subgroup of $\widetilde{T}_i^{(n)}$. Let $\widetilde{T}_*^{(n)}$ be the direct product of $\widetilde{T}_{*,1}^{(n)}, \cdots, \widetilde{T}_{*,k}^{(n)}$ with amalgamated μ_n. Then $\widetilde{T}_*^{(n)}$ is a maximal abelian subgroup of $\widetilde{T}^{(n)}$.

Assume $|n|_F = 1$. Let $\widetilde{T}_{\mathfrak{o}}^{(n)} = \widetilde{T}_{\mathfrak{o}} \cap \widetilde{M}^{(n)}$.

Lemma 2.6.5. *The group $Z_{\widetilde{M}^{(n)}}\widetilde{T}_{\mathfrak{o}}^{(n)}$ is a maximal abelian subgroup of $\widetilde{T}^{(n)}$.*

Let $\widetilde{T}_*$ be a maximal abelian subgroup of $\widetilde{T}$. We again assume $\widetilde{T}_* \cap \widetilde{T}^{(n)}$ is a maximal abelian subgroup of $\widetilde{T}^{(n)}$. When $|n|_F = 1$, the other maximal abelian subgroup we consider is the standard maximal abelian subgroup $\widetilde{T}_*^{\mathrm{st}}$.

Chapter 3

Semi-Whittaker Coefficients: Local Theory

In this section, F is a non-Archimedean local field. Recall that we use $\widetilde{\mathrm{GL}}_r$ to denote $\widetilde{\mathrm{GL}}_r(F)$.

3.1 The principal series representations

The principal series representations of $\widetilde{\mathrm{GL}}_r$ were studied in [KP84]. For the generalization to metaplectic covers of the other reductive groups, see McNamara [McN12].

We start with the representation theory of $\widetilde{T}$. In general, $\widetilde{T}$ is not abelian, but it is a two-step nilpotent group. The irreducible genuine representations of $\widetilde{T}$ are parameterized in the following way ([McN12] Theorem 5.1): start with a genuine character χ on the center of $\widetilde{T}$ and extend it to a character χ' on any maximal abelian subgroup $\widetilde{T}_*$, then the induced representation $i(\chi') := \mathrm{ind}_{\widetilde{T}_*}^{\widetilde{T}} \chi'$ is irreducible. This construction is independent on the choice of $\widetilde{T}_*$ and of the extension of characters.

We extend $i(\chi')$ to a representation $i_{\widetilde{B}}(\chi')$ on $\widetilde{B} = \widetilde{T}U$ by letting U act trivially. Let δ_B be the modular quasicharacter of B. Then $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{GL}}_r} i_{\widetilde{B}}(\chi')\delta_B^{1/2}$ is the principal series representation. This representation is denoted by $I(\chi')$, although its isomorphism class only depends on χ.

There is an alternative way to describe the principal series representations. We can extend the character χ' to $\widetilde{B}_* = \widetilde{T}_*U$, then induce it to $\widetilde{\mathrm{GL}}_r$. The representation $\mathrm{Ind}_{\widetilde{B}_*}^{\widetilde{\mathrm{GL}}_r} \chi'\delta_B^{1/2}$ is isomorphic to $I(\chi')$.

The representation $I(\chi')$ is irreducible when χ is in general position. For a positive root α, there is an embedding $i_\alpha : \mathrm{SL}_2 \to \mathrm{GL}_r$. Define

$$\chi_\alpha^n(t) = \chi\left(i_\alpha \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}^n\right).$$

Theorem 3.1.1. *Suppose that $\chi_\alpha^n \neq |\cdot|_F^{\pm 1}$ for all the positive roots α. Then $I(\chi')$ is irreducible.*

This is proved by the theory of intertwining operators; see [KP84] Corollary I.2.8.

If $\chi_\alpha^n = |\cdot|_F$ for all the positive simple roots α, we call χ *exceptional*. In this case, $I(\chi')$ is reducible, and we are interested in the unique irreducible subquotient of $I(\chi')$. Recall that the intertwining operator $T_w : I(\chi') \to I({}^w\chi')$ is defined as

$$(T_w f)(g) = \int_{U(w)} f(w^{-1}ug)du.$$

where $U(w)$ is the subgroup of U corresponding to roots $\alpha > 0$ such that $w^{-1}\alpha < 0$. If this converges for all $f \in I(\chi')$ and is non-trivial, then it is a generator of $\mathrm{Hom}_{\widetilde{\mathrm{GL}}_r}(I(\chi'), I({}^w\chi'))$. For general χ, the intertwining operator can be defined via analytic continuation.

Theorem 3.1.2. *Let χ be exceptional. Let*

$$\Theta(\chi') = \mathrm{Im}(T_{w_0} : I(\chi') \to I({}^{w_0}\chi')),$$

where w_0 is the longest element of W. Then

(1) *$\Theta(\chi')$ is the unique irreducible subrepresentation of $I({}^{w_0}\chi')$.*

(2) *$\Theta(\chi')$ is the unique irreducible quotient representation of $I(\chi')$.*

(3) *The Jacquet of $\Theta(\chi')$ with respect to U is $J_U(\Theta(\chi')) \cong \mathrm{ind}_{w_0\widetilde{T}_* w_0^{-1}}^{\widetilde{T}}({}^{w_0}\chi'\delta_B^{1/2})$.*

This is [KP84] Theorem I.2.9. $\Theta(\chi')$ is called exceptional.

The Whittaker models of exceptional representations are studied in [KP84] Section I.3. These authors have shown the following results.

Proposition 3.1.3. *Suppose that $|n|_F = 1$.*

(1) *The representation* $\Theta(\chi')$ *has a unique Whittaker model if and only if* $n = r$ *or* $n = r + 1$, *and* $2(c+1) \equiv 0 \mod n$.

(2) *The representation* $\Theta(\chi')$ *does not have a Whittaker model if* $n \leq r - 1$.

(3) *The representation* $\Theta(\chi')$ *has a finite number of independent nonzero Whittaker models if* $n \geq r + 1$.

Remark 3.1.4. In the above proposition, parts (1) and (3) are also true when $|n|_F > 1$. This is shown in [KP84] Section II by using global arguments. Part (2) is expected to be true when $|n|_F > 1$, but this is known only when $n = 2$; see Kaplan [Kapar] Theorem 2.6 and Flicker-Kazhdan-Savin [FKS90].

Remark 3.1.5. When $r = 1$, we take $\Theta(\chi')$ to be $\mathrm{Ind}_{\widetilde{T}_*}^{\widetilde{T}} \chi'$. This fits into the metaplectic tensor product perfectly.

3.2 Restrictions

We study the restriction functor $\mathrm{Res}_{\widetilde{\mathrm{GL}}_r^{(n)}}^{\widetilde{\mathrm{GL}}_r}$ in this section. We obtain an explicit description of the restriction of the principal series representations and exceptional representations from $\widetilde{\mathrm{GL}}_r$ to $\widetilde{\mathrm{GL}}_r^{(n)}$. This is useful in Section 3.5 where we give explicit examples of the metaplectic tensor product.

Notice that $\widetilde{\mathrm{GL}}_r^{(n)}$ is an open normal subgroup of $\widetilde{\mathrm{GL}}_r$, and $\widetilde{\mathrm{GL}}_r/\widetilde{\mathrm{GL}}_r^{(n)} \cong F^\times/F^{\times n}$ is finite and abelian. By Gelbart-Knapp [GK82] Lemma 2.1, if $I(\chi')$ is irreducible, and π is an irreducible constituent of $I(\chi')|_{\widetilde{\mathrm{GL}}_r^{(n)}}$, then

$$I(\chi')|_{\widetilde{\mathrm{GL}}^{(n)}} = \sum_g m\ {}^g\pi.$$

The multiplicities m depend only on $I(\chi')$, and the direct sum is over certain elements of $\widetilde{\mathrm{GL}}_r$.

From now on, we assume $\widetilde{T}_*^{(n)} := \widetilde{T}_* \cap \widetilde{T}^{(n)}$ is a maximal abelian subgroup of $\widetilde{T}^{(n)}$. Let $\widetilde{B}_* = \widetilde{T}_* U$ and $\widetilde{B}_*^{(n)} = \widetilde{T}_*^{(n)} U$.

Proposition 3.2.1.

$$I(\chi')|_{\widetilde{\mathrm{GL}}_r^{(n)}} \cong \bigoplus_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T}/\widetilde{T}_*} \mathrm{Ind}_{{}^x\widetilde{B}_*^{(n)}}^{\widetilde{\mathrm{GL}}_r^{(n)}} ({}^x\chi' \delta_B^{1/2})|_{{}^x\widetilde{B}_*^{(n)}}. \tag{3.1}$$

Proof. This follows from Bernstein-Zelevinsky [BZ77] Theorem 5.2. We are working with representations of $\widetilde{\mathrm{GL}}_r$. Let us choose triples $\widetilde{B}, \widetilde{T}, U$ with trivial character on U on the induced functor side, and $\widetilde{\mathrm{GL}}_r^{(n)}, \widetilde{\mathrm{GL}}_r^{(n)}, \{1\}$. with trivial character on $\{1\}$ on the Jacquet functor side. The Jacquet functor in this case is the restriction functor.

The resulting functor is glued by functors indexed by the double coset space $\widetilde{\mathrm{GL}}_r^{(n)} \backslash \widetilde{\mathrm{GL}}_r / \widetilde{B}$. This double coset space is a singleton since $\widetilde{T}\widetilde{\mathrm{GL}}_r^{(n)} = \widetilde{\mathrm{GL}}_r$. Therefore, the functor is the composition of the induction functor from $\widetilde{T} \cap \widetilde{\mathrm{GL}}_r^{(n)}$ to $\widetilde{\mathrm{GL}}_r^{(n)}$ and the restriction functor $\mathrm{Res}^{\widetilde{T}}_{\widetilde{T}^{(n)}}$.

By [GK82] Lemma 2.1, $\mathrm{ind}^{\widetilde{T}}_{\widetilde{T}_*} \chi'|_{\widetilde{T}^{(n)}}$ is a direct sum of irreducible $\widetilde{T}^{(n)}$-representations. On the other hand, it has a Jordan-Holder series whose composition factors are

$$\mathrm{ind}^{\widetilde{T}^{(n)}}_{{}^x\widetilde{T}_*^{(n)}} {}^x\chi', \qquad x \in \widetilde{T}^{(n)} \backslash \widetilde{T} / \widetilde{T}_*.$$

Notice that $\widetilde{T}^{(n)}$ is also a Heisenberg group and $\widetilde{T}^{(n)} \cap {}^x\widetilde{T}_* = {}^x(\widetilde{T}_*^{(n)})$ is again a maximal abelian subgroup of $\widetilde{T}^{(n)}$. This implies $\mathrm{ind}^{\widetilde{T}^{(n)}}_{{}^x\widetilde{T}_*^{(n)}} {}^x\chi'$ is irreducible. Thus,

$$(\mathrm{ind}^{\widetilde{T}}_{\widetilde{T}_*} \chi')|_{\widetilde{T}^{(n)}} = \bigoplus_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T} / \widetilde{T}_*} \mathrm{ind}^{\widetilde{T}^{(n)}}_{{}^x\widetilde{T}_*^{(n)}} {}^x\chi'.$$

Now the proposition follows.

□

Remark 3.2.2. Notice that Eq. (3.1) depends on the choice of maximal abelian subgroup. Indeed, when χ is in general position, the condition that $\widetilde{T}_* \cap \widetilde{T}^{(n)} = \widetilde{T}_*^{(n)}$ implies each component is irreducible. Without this condition, we get a similar decomposition, but the components are reducible.

Next we show that, when χ is in general position, the components in Proposition 3.2.1 are irreducible. Let us write $V(\chi') = \mathrm{Ind}^{\widetilde{\mathrm{GL}}_r^{(n)}}_{\widetilde{B}_*^{(n)}} (\chi' \delta_B^{1/2})|_{\widetilde{B}_*^{(n)}}$. Thus Proposition 3.2.1 becomes

$$I(\chi')|_{\widetilde{\mathrm{GL}}_r^{(n)}} \cong \bigoplus_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T} / \widetilde{T}_*} V({}^x\chi').$$

Definition 3.2.3. A character of $Z_{\widetilde{\mathrm{GL}}_r} \widetilde{T}^n$ or $Z_{\widetilde{\mathrm{GL}}_r^{(n)}} \widetilde{T}^n$ is called regular if ${}^w\chi \neq \chi$ for all $w \in W, w \neq I$.

Lemma 3.2.4.

(1) *The $\widetilde{T}^{(n)}$-module $J_U(V(\chi'))$ has a Jordan-Holder series whose composition factors are*

$$\mathrm{ind}_{w\widetilde{T}_*^{(n)}w^{-1}}^{\widetilde{T}^{(n)}}({}^w\chi'\delta_B^{1/2})(w \in W).$$

(2) *If χ is regular, then for any extension χ', $\chi'|_{Z_{\widetilde{\mathrm{GL}}_r^{(n)}}\widetilde{T}^n}$ is regular. Moreover,*

$$J_U(V(\chi')) \cong \bigoplus_{w\in W} \mathrm{ind}_{w\widetilde{T}_*^{(n)}w^{-1}}^{\widetilde{T}^{(n)}}({}^w\chi'\delta_B^{1/2}).$$

Proof. The first part follows from [BZ77] Theorem 5.2. For the second part, we only need to show that $\chi'|_{Z_{\widetilde{\mathrm{GL}}_r^{(n)}}\widetilde{T}^n}$ is regular. Indeed, if χ is regular, then for any $w \in W$, there exists $x \in Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}^n$ such that $\chi(w^{-1}xw) \neq \chi(x)$. Without loss of generality, we may assume $x \in \widetilde{T}^n$. This implies that $\chi'|_{Z_{\widetilde{\mathrm{GL}}_r^{(n)}}\widetilde{T}^n}$ is regular, for any extension χ' of χ. □

Lemma 3.2.5. *Let χ_1, χ_2 be two quasicharacters of $Z_{\widetilde{\mathrm{GL}}_r^{(n)}}\widetilde{T}^n$ and let χ_1', χ_2' be extensions to $\widetilde{T}_*^{(n)}$. Suppose χ_1 is regular. Then*

$$\dim \mathrm{Hom}_{\widetilde{\mathrm{GL}}_r^{(n)}}(V(\chi_1'), V(\chi_2')) \leq 1.$$

The equality holds if and only if $\chi_2 = {}^w\chi_1$ for some $w \in W$.

Proof. This is an immediate application of Lemma 3.2.4, Frobenius reciprocity, and the fact that $\mathrm{ind}_{w\widetilde{T}_*^{(n)}w^{-1}}^{\widetilde{T}^{(n)}}({}^w\chi'\delta_B^{1/2})$ is irreducible. □

We now restrict the intertwining operator $T_w : I(\chi') \to I({}^w\chi')$ to Eq. (3.1). It gives

$$T_w : V({}^x\chi') \to V({}^{wx}\chi').$$

Proposition 3.2.6. *If $\chi_\alpha^n \neq |\cdot|^{\pm 1}$, for all positive roots α, then $V(\chi')$ is irreducible.*

Proof. Under the assumption, $T_w : I(\chi') \to I({}^w\chi')$ is an isomorphism, its restriction

$$T_w : V(\chi') \to V({}^w\chi')$$

is again an isomorphism. Arguing as in [KP84] Corollary I.2.8, we can show that $V(\chi')$ is irreducible. □

Similarly we can deduce results for exceptional representations.

Theorem 3.2.7. *Let χ be exceptional. Let*

$$V_0(\chi') = \mathrm{Im}(T_{w_0} : V(\chi') \to V({}^{w_0}\chi')),$$

where w_0 is the longest elements of W. Then

(1) *$V_0(\chi')$ is the unique irreducible subrepresentation of $V({}^{w_0}\chi')$.*

(2) *$V_0(\chi')$ is the unique irreducible quotient representation of $V(\chi')$.*

(3) $J_U(V_0(\chi')) \cong \mathrm{ind}_{w_0\widetilde{T}_*^{(n)}w_0^{-1}}^{\widetilde{T}^{(n)}}({}^{w_0}\chi'\delta_B^{1/2})$.

Proof. The map

$$T_{w_0} : I(\chi') \to I({}^{w_0}\chi')$$

restricts to

$$T_{w_0} : \bigoplus_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T}/\widetilde{T}_*} V({}^{x}\chi') \to \bigoplus_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T}/\widetilde{T}_*} V({}^{w_0 x}\chi').$$

This implies that

$$\Theta(\chi')|_{\widetilde{\mathrm{GL}}_r^{(n)}} = \bigoplus_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T}/\widetilde{T}_*} V_0({}^{x}\chi').$$

We first show part (3). From the exactness of the Jacquet functor, $J_U(V_0(\chi'))$ is a subrepresentation of both $J_U(V(\chi'))$ and $J_U(\Theta(\chi'))$. Therefore, $J_U(V_0(\chi')) \cong \mathrm{ind}_{w_0\widetilde{T}_*^{(n)}w_0^{-1}}^{\widetilde{T}^{(n)}}({}^{w_0}\chi'\delta_B^{1/2})$.

The representation $\Theta(\chi')|_{\widetilde{\mathrm{GL}}_r^{(n)}}$ is a direct sum of irreducible constituents, which are conjugate to each other. Thus $V_0(\chi')$ is a direct sum of some of these components. This implies that $J_U(V_0(\chi'))$ is also a direct sum of the corresponding Jacquet modules which are conjugate to each other. Thus $V_0(\chi')$ is irreducible since $J_U(V_0(\chi'))$ is irreducible.

If π is another irreducible quotient representation of $V(\chi')$, then its Jacquet module is a quotient of $J_U(V(\chi'))$, hence there is a nonzero homomorphism $J_U(\pi) \to \mathrm{ind}_{w\widetilde{T}_*^{(n)}w^{-1}}^{\widetilde{T}^{(n)}}({}^{w}\chi'\delta_B^{1/2})$ for some $w \in W$. By Frobenius reciprocity, there is a nonzero intertwining map $\pi \to V({}^{w}\chi')$. The composition

$$V(\chi') \to \pi \to V({}^{w}\chi')$$

is nonzero and it must be a constant multiple of T_w. Therefore, the composition

$$V(\chi') \to \pi \to V({}^{w}\chi') \xrightarrow{T_{w_0 w^{-1}}} V({}^{w_0}\chi')$$

is T_{w_0} and its image is $V_0(\chi')$. We see that $V_0(\chi')$ is a quotient of π, and since π is irreducible, they must be the same. This proves part (2). Part (1) follows from part (2) by duality.

□

As a corollary, we describe the decomposition of $\Theta(\chi')$ when restricted to $\widetilde{\mathrm{GL}}_r^{(n)}$.

Corollary 3.2.8.

$$\Theta(\chi')|_{\widetilde{\mathrm{GL}}_r^{(n)}} \cong \bigoplus_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T} / \widetilde{T}_*} V_0({}^{x}\chi').$$

3.3 Principal series of Levi subgroups

Let λ be a partition of r and write $\widetilde{M}$ for $\widetilde{M}_\lambda$. The principal series representations and exceptional representations can be similarly defined on $\widetilde{M}$. Recall we may identify GL_{r_i} as a subgroup of M via the embedding

$$g_i \mapsto \mathrm{diag}(I_{r_1}, \cdots, g_i, \cdots, I_{r_k}).$$

Let B_i be the standard Borel subgroup of GL_{r_i} and δ_{B_i} be the modular quasicharacter of B_i in GL_{r_i}.

Let χ be a genuine character of $Z_{\widetilde{\mathrm{GL}}_r} \widetilde{T}^n$, extend it to a character χ' of $\widetilde{T}_*$. The genuine representation $\pi_{\widetilde{T}}(\chi') := \mathrm{ind}_{\widetilde{T}_*}^{\widetilde{T}} \chi'$ is irreducible. The principal series representation $I(\chi')$ is the induced representation $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{M}} \pi_{\widetilde{T}}(\chi') \otimes \delta_M^{1/2}$, where $\delta_M = \delta_{B_1} \otimes \cdots \otimes \delta_{B_k}$. There is an alternative way to describe it as in the general linear case.

The theory of intertwining operators applies just as the general linear case. Therefore, $I(\chi')$ is irreducible when χ is in general position.

Theorem 3.3.1. *Suppose that $\chi_\alpha^n \neq |\cdot|_F^{\pm 1}$ for all the positive roots α in M. Then $I(\chi')$ is irreducible.*

If $\chi_\alpha^n = |\cdot|_F$ for all positive simple roots α in M, we call it *exceptional.*

Theorem 3.3.2. *Let χ be exceptional. Let*

$$\Theta(\chi') = \mathrm{Im}(T_{w_{M,0}} : I(\chi') \to I({}^{w_{M,0}}\chi')),$$

where $w_{M,0}$ is the longest element of $W(M)$. Then

(1) *$\Theta(\chi')$ is the unique irreducible subrepresentation of $I({}^{w_{M,0}}\chi')$.*

(2) *$\Theta(\chi')$ is the unique irreducible quotient representation of $I(\chi')$.*

(3) *$J_{U\cap M}(\Theta(\chi')) \cong \mathrm{ind}_{w_{M,0}\widetilde{T}_* w_{M,0}^{-1}}^{\widetilde{T}}({}^{w_{M,0}}\chi'\delta_M^{1/2})$.*

We also want to study $I(\chi')|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}$, and $\Theta(\chi')|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}$. The arguments in Section 3.2 apply in this case without essential change. We only state the results here.

Proposition 3.3.3.

$$I(\chi')|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}} \cong \bigoplus_{x\in Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}^{(n)}\backslash\widetilde{T}/\widetilde{T}_*} \mathrm{Ind}_{{}^x(Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}\cap\widetilde{B}_*)}^{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}} {}^x\chi'\delta_M^{1/2}.$$

Proposition 3.3.4. *If $\chi_\alpha^n \neq |\cdot|^{\pm 1}$ for all positive roots α in M, then $\mathrm{Ind}_{{}^x(Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}\cap\widetilde{B}_*)}^{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}} {}^x\chi'\delta_M^{1/2}$ is irreducible.*

As in the general linear case, write $V({}^x\chi') = \mathrm{Ind}_{{}^x(Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}\cap\widetilde{B}_*)}^{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}} {}^x\chi'\delta_M^{1/2}$.

Proposition 3.3.5. *Let χ be exceptional. Let*

$$V_0(\chi') = \mathrm{Im}(T_{w_{M,0}} : V(\chi') \to V({}^{w_{M,0}}\chi')),$$

where $w_{M,0}$ is the longest elements of $W(M)$. Then

(1) *$V_0(\chi')$ is the unique irreducible subrepresentation of $V({}^{w_{M,0}}\chi')$.*

(2) *$V_0(\chi')$ is the unique irreducible quotient representation of $V(\chi')$.*

(3) *$J_{U\cap M}(V_0(\chi')) \cong \mathrm{ind}_{Z_{\widetilde{\mathrm{GL}}_r} w_{M,0}\widetilde{T}_*^{(n)} w_{M,0}^{-1}}^{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}^{(n)}}({}^{w_{M,0}}\chi'\delta_M^{1/2})$.*

Proposition 3.3.6.

$$\Theta(\chi')|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}} \cong \bigoplus_{x\in Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}^{(n)}\backslash\widetilde{T}/\widetilde{T}_*} V_0({}^x\chi').$$

Lastly, let χ' be an exceptional character for $\widetilde{\mathrm{GL}}_r$. Let P be the parabolic subgroup of GL_r with Levi subgroup M, and R be its unipotent radical. Let δ_P be the modular quasicharacter of GL_r with respect to P. Recall we have $\delta_M \cdot \delta_P = \delta_{\mathrm{GL}_r}$ and $w_0 = w_{M,0}w^M$.

Proposition 3.3.7. *The character* ${}^{w^M}\chi' \cdot \delta_P^{1/2}$ *is exceptional for* M, *and*

$$J_R(\Theta_{\widetilde{\mathrm{GL}}_r}(\chi')) \cong \Theta_{\widetilde{M}}({}^{w^M}\chi' \cdot \delta_P^{1/2}).$$

Proof. The Weyl element w^M permutes blocks of M, thus the character ${}^{w^M}\chi' \cdot \delta_P^{1/2}$ is exceptional for $\widetilde{M}$. To prove the isomorphism of twisted Jacquet modules, we apply $J_{U\cap M}(-)$ on both sides. The left-hand side is

$$J_{U\cap M}(J_R(\Theta_{\widetilde{\mathrm{GL}}_r}(\chi'))) = J_U(\Theta_{\widetilde{\mathrm{GL}}_r}(\chi')) \cong \mathrm{ind}_{w_0\widetilde{T}_* w_0^{-1}}^{\widetilde{T}}({}^{w_0}\chi' \delta_{\mathrm{GL}_r}^{1/2});$$

while the right-hand side is

$$J_{U\cap M}(\Theta_{\widetilde{M}}({}^{w^M}\chi' \cdot \delta_P^{1/2})) \cong \mathrm{ind}_{w_{M,0}\widetilde{T}_* w_{M,0}^{-1}}\, {}^{w_0}\chi' \cdot \delta_P^{1/2}\delta_M^{1/2} \cong \mathrm{ind}_{w_0\widetilde{T}_* w_0^{-1}}^{\widetilde{T}}({}^{w_0}\chi' \delta_{\mathrm{GL}_r}^{1/2}).$$

This implies that $J_R(\Theta_{\widetilde{\mathrm{GL}}_r}(\chi'))$ and $\Theta_{\widetilde{M}}({}^{w^M}\chi' \cdot \delta_P^{1/2})$ are both irreducible subrepresentations of $I({}^{w^M}\chi' \cdot \delta_P^{1/2})$. Thus they are isomorphic.

□

3.4 The metaplectic tensor product

One of the basic constructions in the representation theory of $\mathrm{GL}_r(F)$ is parabolic induction. Let $r = r_1 + \cdots + r_k$ be a partition of r, and let $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$ be a Levi subgroup. We start with a list of representations, one for each of $\mathrm{GL}_{r_1}, \cdots, \mathrm{GL}_{r_n}$, and then form their tensor product to obtain a representation of M. However, $\widetilde{M}$ is not simply the amalgamated direct product of the various $\widetilde{\mathrm{GL}}_{r_i}$, this construction cannot be generalized directly to the metaplectic case. Fortunately, we have a replacement, which is defined in Mezo [Mez04]. We review the construction in this section. The two-fold cover case was outlined in Bump and Ginzburg [BG92], and studied in full detail in Kable [Kab01]. For the global setup and further properties see Takeda [Tak16, Takar].

We observe that any element $m \in \widetilde{M}$ may be written as a product $g_1 \cdots g_k$, such that $\mathbf{p}(g_i) \in \mathrm{GL}_{r_i}$ for $1 \leq i \leq k$. Recall

$$\widetilde{M}^{(n)} = \{m \in \widetilde{M} : \det g_1, \cdots, \det g_k \in F^{\times n}\}$$

and $\widetilde{\mathrm{GL}}_{r_i}^{(n)} = \widetilde{M}^{(n)} \cap \widetilde{\mathrm{GL}}_{r_i}$.

Let $\pi_1, \cdots, \pi_k$ be irreducible genuine representations of $\widetilde{\mathrm{GL}}_{r_1}, \cdots, \widetilde{\mathrm{GL}}_{r_k}$, respectively. The construction of the metaplectic tensor product takes several steps.

First of all, for each i, fix an irreducible constituent $\pi_i^{(n)}$ of the restriction $\pi_i \mid_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}}$ of π_i to $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$. Then we have

$$\pi_i|_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}} = \sum_g m_i \, {}^g(\pi_i^{(n)}),$$

where g runs through a finite subset of $\widetilde{\mathrm{GL}}_{r_i}$, m_i is a positive multiplicity and ${}^g(\pi_i^{(n)})$ is the representation twisted by g. Then we construct the tensor product representation

$$\pi_1^{(n)} \otimes \cdots \otimes \pi_k^{(n)}$$

of the group $\widetilde{\mathrm{GL}}_{r_1}^{(n)} \otimes \cdots \otimes \widetilde{\mathrm{GL}}_{r_k}^{(n)}$. Because the representations $\pi_1, \cdots, \pi_k$ are genuine, this tensor product representation descends to a representation of the group $\widetilde{M}^{(n)}$, i.e. the representation factors through the natural surjection

$$\widetilde{\mathrm{GL}}_{r_1}^{(n)} \times \cdots \times \widetilde{\mathrm{GL}}_{r_k}^{(n)} \twoheadrightarrow \widetilde{M}^{(n)}.$$

We denote this representation of $\widetilde{M}^{(n)}$ by

$$\pi^{(n)} := \pi_1^{(n)} \tilde{\otimes} \cdots \tilde{\otimes} \pi_k^{(n)},$$

and call it the metaplectic tensor product of $\pi_1^{(n)}, \cdots, \pi_k^{(n)}$.

Let ω be a character on the center $Z_{\widetilde{\mathrm{GL}}_r}$ such that for all $(aI_r, \zeta) \in Z_{\widetilde{\mathrm{GL}}(r)} \cap \widetilde{M}^{(n)}$ where $a \in F^\times$, we have

$$\omega(aI_r, \zeta) = \pi^{(n)}(aI_r, \zeta) = \zeta \pi_1^{(n)}(aI_{r_1}, 1) \cdots \pi_r^{(n)}(aI_{r_k}, 1).$$

Namely, ω agrees with $\pi^{(n)}$ on the intersection $Z_{\widetilde{\mathrm{GL}}_r} \cap \widetilde{M}^{(n)}$. We can extend $\pi^{(n)}$ to the representation

$$\pi_\omega^{(n)} := \omega \pi^{(n)}$$

of $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$ by letting $Z_{\widetilde{\mathrm{GL}}_r}$ act by ω.

The last step is crucial. If we induce $\pi_\omega^{(n)}$ to $\widetilde{M}$, the resulting representation is usually reducible. To get an irreducible representation, we extend the representation $\pi_\omega^{(n)}$ to a representation ρ_ω of a subgroup $\widetilde{H}$ of $\widetilde{M}$ so that ρ_ω satisfies Mackey's irreducibility criterion and the induced representation

$$\pi_\omega := \mathrm{Ind}_{\widetilde{H}}^{\widetilde{M}}\rho_\omega$$

is irreducible. It is always possible to find such $\widetilde{H}$ and moreover $\widetilde{H}$ can be chosen to be normal. The construction of π_ω is independent of the choices of $\pi_i^{(n)}$, $\widetilde{H}$ and ρ_ω, and it only depends on ω (see [Mez04] Section 4).

We write

$$\pi_\omega = (\pi_1\tilde{\otimes}\cdots\tilde{\otimes}\pi_k)_\omega$$

and call it the metaplectic tensor product of $\pi_1, \cdots, \pi_k$ with the character ω.

The metaplectic tensor product π_ω is unique up to twist.

Proposition 3.4.1 ([Mez04] Lemma 5.1). *Let*

$$\pi_1, \cdots, \pi_k \qquad \textit{and} \qquad \pi_1', \cdots, \pi_k'$$

be genuine representations of $\widetilde{\mathrm{GL}}_{r_1}, \cdots, \widetilde{\mathrm{GL}}_{r_k}$. They give rise to isomorphic metaplectic tensor products with a character ω, i.e.

$$(\pi_1\tilde{\otimes}\cdots\tilde{\otimes}\pi_k)_\omega \cong (\pi_1'\tilde{\otimes}\cdots\tilde{\otimes}\pi_k')_\omega$$

if and only for each i there exists a character ω_i of $\widetilde{\mathrm{GL}}_{r_i}$, trivial on $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$, such that $\pi_i \cong \omega_i \otimes \pi_i'$.

Remark 3.4.2. Notice that the metaplectic tensor product generally depends on the choice of ω. If the center $Z_{\widetilde{\mathrm{GL}}_r}$ is already contained in $\widetilde{M}^{(n)}$, we have $\pi_\omega^{(n)} = \pi^{(n)}$ and hence there is no actual choice for ω and the metaplectic tensor product is canonical. This is the case, for example, when $n = 2$ or $n \mid r$.

A representation of $\widetilde{M}$ is always a metaplectic tensor product ([Tak16], Lemma 4.5). Moreover, we have the following useful lemmas.

Lemma 3.4.3 ([Tak16] Lemma 4.6). *Let π and π' be irreducible admissible representations of $\widetilde{M}$. Then π and π' are equivalent if and only if $\pi|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}$ and $\pi'|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}$ have an equivalent constituent.*

Lemma 3.4.4 ([Tak16] Proposition 4.7)**.** *We have*

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{M}}\pi_\omega^{(n)} = m\pi_\omega$$

for some finite multiplicity m, so every constituent of $\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}^{\widetilde{M}}\pi_\omega^{(n)} = m\pi_\omega$ is isomorphic to π_ω.

Indeed, we can verify that $m = [\widetilde{H} : Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}]$.

Lemma 3.4.5. *We have*

$$\mathrm{Ind}_{\widetilde{M}^{(n)}}^{\widetilde{M}}\pi^{(n)} = m\left(\bigoplus_\xi \pi_\xi\right)$$

where m is $[\widetilde{H} : Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}]$ and ξ is over the finite set of characters of $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$ that are trivial on $\widetilde{M}^{(n)}$.

Proof. The proof is the same as in [Tak16] Proposition 4.7; see also [Takar] Proposition 3.2. □

3.5 Examples

We give some examples of the metaplectic tensor product in this section. The key ingredient in the proof is Lemma 3.4.3. This allows us to compare irreducible smooth representations of $\widetilde{M}$ by restricting to $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$.

Let χ be a genuine quasicharacter on $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}^n$, and $\omega = \chi|_{Z_{\widetilde{\mathrm{GL}}_r}}$ be the central quasicharacter. For each i, let $\widetilde{T}_{*,i}^{(n)}$ be a maximal abelian subgroup of $\widetilde{T}_i^{(n)}$. Let $\widetilde{T}_*^{(n)}$ be the direct product of $\widetilde{T}_{*,1}^{(n)}, \cdots, \widetilde{T}_{*,k}^{(n)}$ with amalgamated μ_n. Then $\widetilde{T}_*^{(n)}$ is a maximal abelian subgroup of $\widetilde{T}^{(n)}$. Let $\widetilde{T}_*$ be a maximal abelian subgroup of $\widetilde{T}$ such that $\widetilde{T} \cap \widetilde{T}^{(n)} = \widetilde{T}_*^{(n)}$.

Let χ' be an extension of χ to $\widetilde{T}_*$. We may decompose $\chi'|_{\widetilde{T}_*^{(n)}}$ as

$$\chi_1 \tilde{\otimes} \cdots \tilde{\otimes} \chi_k,$$

where χ_i is a genuine character on $\widetilde{T}_{*,i}^{(n)}$. Let $\widetilde{T}_{*,i}$ be a maximal abelian subgroup of $\widetilde{T}_i$ such that $\widetilde{T}_i^{(n)} \cap \widetilde{T}_{*,i} = \widetilde{T}_{*,i}^{(n)}$. We still use χ_i to denote an extension of χ_i to $\widetilde{T}_{*,i}$ (this extension is not unique). When χ is in general position, so are χ_i's. Therefore the principal series representations $I(\chi_i')$ on $\widetilde{\mathrm{GL}}_{r_i}$ are irreducible.

Theorem 3.5.1. *Assume that χ is in general position. Then the metaplectic tensor product $(I(\chi_1')\tilde{\otimes}\cdots\tilde{\otimes}I(\chi_k'))_\omega$ is independent on the choices of χ_i. Moreover, as representations of $\widetilde{M}$,*

$$I(\chi') \cong (I(\chi_1')\tilde{\otimes}\cdots\tilde{\otimes}I(\chi_k'))_\omega$$

This result shows that, for principal series representations, the metaplectic tensor product can be viewed as an instance of Langlands functoriality on covering groups; see Gan [Gan16].

Proof. Indeed, the choice of the character χ_i on $\widetilde{T}_{*,i}$ is up to a character of $\widetilde{T}_{*,i}/\widetilde{T}_{*,i}^{(n)}$. Thus the resulting principal series representations differ by a character that is trivial on $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$. By Proposition 3.4.1, the metaplectic tensor products are still in the same isomorphism class. This proves the well-definedness.

For the second assertion, let us follow the construction of metaplectic tensor product. For $I(\chi_i')|_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}}$, we choose one irreducible constituent $\mathrm{Ind}_{\widetilde{B}_{*,i}^{(n)}}^{\widetilde{\mathrm{GL}}_{r_i}^{(n)}} \chi_i'\delta_{B_i}^{1/2}$. Then as representations of $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$,

$$\omega(\mathrm{Ind}_{\widetilde{B}_{*,1}^{(n)}}^{\widetilde{\mathrm{GL}}_{r_1}^{(n)}} \chi_1'\delta_{B_1}^{1/2}\tilde{\otimes}\cdots\tilde{\otimes}\mathrm{Ind}_{\widetilde{B}_{*,k}^{(n)}}^{\widetilde{\mathrm{GL}}_{r_k}^{(n)}} \chi_k'\delta_{B_k}^{1/2}) \cong \omega\mathrm{Ind}_{\widetilde{B}_*^{(n)}}^{\widetilde{M}^{(n)}}\chi'\delta_M^{1/2}.$$

This is an irreducible constituent of

$$(I(\chi_1')\tilde{\otimes}\cdots\tilde{\otimes}I(\chi_k'))_\omega|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}.$$

On the other hand,

$$\omega\mathrm{Ind}_{\widetilde{B}_*^{(n)}}^{\widetilde{M}^{(n)}}\chi'\delta_M^{1/2} \cong \mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{B}_*^{(n)}}^{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}\chi'\delta_M^{1/2}.$$

is also an irreducible constituent of $I(\chi')|_{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}}$. By Lemma 3.4.3, we are done. □

Next, we turn to exceptional representations. We start with an exceptional character χ on $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}^n$, and form the exceptional representation $\Theta_{\widetilde{M}}(\chi')$ as the irreducible quotient of $\mathrm{Ind}_{\widetilde{B}_*}^{\widetilde{M}}\chi'\delta_M^{1/2}$. The characters χ_i's are defined as in the previous case.

Theorem 3.5.2. *The metaplectic tensor product $(\Theta(\chi_1')\tilde{\otimes}\cdots\tilde{\otimes}\Theta(\chi_k'))_\omega$ is well-defined. As representations of $\widetilde{M}$,*

$$\Theta_{\widetilde{M}}(\chi') \cong (\Theta(\chi_1')\tilde{\otimes}\cdots\tilde{\otimes}\Theta(\chi_k'))_\omega.$$

Proof. Again, we want to show that both sides have an equivalent irreducible constituent when restricted to $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$. For the left-hand side, we choose $V_0(\chi'|_{\widetilde{T}_*^{(n)}})$. This is the unique irreducible subrepresentation of $\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r} w_{M,0}\widetilde{B}_*^{(n)} w_{M,0}^{-1}}^{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}} {}^{w_{M,0}}\chi'\delta_M^{1/2}$. The Jacquet module of $V_0(\chi'|_{\widetilde{T}_*^{(n)}})$ is

$$J_{U\cap M}(V_0(\chi'|_{\widetilde{T}_*^{(n)}})) \cong \mathrm{ind}_{Z_{\widetilde{\mathrm{GL}}_r} w_{M,0}\widetilde{T}_*^{(n)} w_{M,0}^{-1}}^{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}^{(n)}}({}^{w_{M,0}}\chi'\delta_M^{1/2}).$$

On the right-hand side, we choose $\omega(V_0(\chi_1')\tilde{\otimes}\cdots\tilde{\otimes}V_0(\chi_k'))$, whose Jacquet module is

$$\begin{aligned}&\omega(\mathrm{ind}_{w_{\mathrm{GL}_{r_1},0}(\widetilde{T}_{1,*}^{(n)})w_{\mathrm{GL}_{r_1},0}^{-1}}^{\widetilde{T}_1^{(n)}}({}^{w_{\mathrm{GL}_{r_1},0}}\chi_1\delta_{B_1}^{1/2})\tilde{\otimes}\cdots\tilde{\otimes}\mathrm{ind}_{w_{\mathrm{GL}_{r_k},0}(\widetilde{T}_{k,*}^{(n)})w_{\mathrm{GL}_{r_k},0,}^{-1}}^{\widetilde{T}_i^{(n)}}({}^{w_{\mathrm{GL}_{r_k},0}}\chi_k\delta_{B_k}^{1/2}))\\ \cong&\mathrm{ind}_{Z_{\widetilde{\mathrm{GL}}_r} w_{M,0}\widetilde{T}_*^{(n)} w_{M,0}^{-1}}^{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{T}^{(n)}}({}^{w_{M,0}}\chi'\delta_M^{1/2}).\end{aligned}$$

Thus $\omega(V_0(\chi_1')\tilde{\otimes}\cdots\tilde{\otimes}V_0(\chi_k'))$ can be also realized as the unique irreducible subrepresentation of

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r} w_{M,0}\widetilde{B}_*^{(n)} w_{M,0}^{-1}}^{Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}} {}^{w_{M,0}}\chi'\delta_M^{1/2}.$$

Therefore, as representations of $Z_{\widetilde{\mathrm{GL}}_r}\widetilde{M}^{(n)}$,

$$V_0(\chi'|_{\widetilde{T}_*^{(n)}}) \cong \omega(V_0(\chi_1')\tilde{\otimes}\cdots\tilde{\otimes}V_0(\chi_k')).$$

By Lemma 3.4.3, we are done. □

Example 3.5.3. Consider the partition (1^r). In this case, $\widetilde{M}$ is just $\widetilde{T}$ and the metaplectic tensor product is just the representation theory of $\widetilde{T}$. The exceptional representation on $\widetilde{\mathrm{GL}}_1$ is $\mathrm{Ind}_A^{\widetilde{\mathrm{GL}}_1}\chi'$, where A is a maximal abelian subgroup of $\widetilde{\mathrm{GL}}_1$, and χ' is an extension of $\chi : F^{\times n} \to \mathbb{C}^\times$ to A. Notice that χ as an irreducible constituent of $\left(\mathrm{Ind}_A^{\widetilde{\mathrm{GL}}_1}\chi'\right)\Big|_{F^{\times n}}$. Let $\chi_1, \cdots, \chi_r$ be characters of $F^{\times n}$. Thus the metaplectic tensor product of $\mathrm{Ind}_A^{\widetilde{\mathrm{GL}}_1}\chi_1', \cdots, \mathrm{Ind}_A^{\widetilde{\mathrm{GL}}_1}\chi_r'$ is $\mathrm{Ind}_{\widetilde{T}_*}^{\widetilde{T}}(\chi_1\otimes\cdots\otimes\chi_r)'$, where $(\chi_1\otimes\cdots\otimes\chi_k)'$ is an extension of $\chi_1\otimes\cdots\otimes\chi_k$ to $\widetilde{T}_*$.

3.6 Semi-Whittaker functionals

Fix an nontrivial additive character $\psi : F \to \mathbb{C}^\times$. For a partition λ of r, let $M = M_\lambda$ be the corresponding Levi subgroup of GL_r. We define a character

$$\psi_\lambda : U_M = U \cap M \to U_M/[U_M, U_M] \to \mathbb{C}^\times$$

as follows. If α is a positive simple root in U_M, then $\psi_\lambda(x_\alpha(a)) = \psi(a)$. We extend this character to $\psi_\lambda : U \to \mathbb{C}^\times$ via the naive projection $U \to U \cap M$. Notice this character agrees with the character defined in Section 2.1. For a smooth representation (π, V) of $\widetilde{\mathrm{GL}}_r$, a linear functional $L : V \to \mathbb{C}$ is called a *λ-semi-Whittaker functional* if $L(\pi(u)v) = \psi_\lambda(u)L(v)$ for all $u \in U, v \in V$. When λ is fixed, we simply say semi-Whittaker functional.

We study semi-Whittaker functionals of exceptional representations. First, we have the following observation for Whittaker functionals of exceptional representations on $\widetilde{\mathrm{GL}}_r$.

Let $\Theta(\chi')$ be an exceptional representation of $\widetilde{\mathrm{GL}}_r$, and $\psi_{Wh} : U \to \mathbb{C}^\times$ such that $\psi_{Wh}(u) = \psi(\sum_{i=1}^{r-1} u_{i,i+1})$. Let $d = \dim J_{U,\psi_{Wh}}(\Theta(\chi'))$. If we restrict $\Theta(\chi')$ to $\widetilde{\mathrm{GL}}_r^{(n)}$, we still have $d = \dim J_{U,\psi_{Wh}}(\Theta(\chi')|_{\widetilde{\mathrm{GL}}_r^{(n)}})$. By the exactness of Jacquet functor and Corollary 3.2.8,

$$\begin{aligned} d &= \sum_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T} / \widetilde{T}_*} \dim J_{U,\psi_{Wh}}(V_0({}^x\chi')) \\ &= \sum_{x \in \widetilde{T}^{(n)} \widetilde{T}_* \backslash \widetilde{T}} \dim J_{U,{}^x\psi_{Wh}}(V_0(\chi')). \end{aligned}$$

Therefore,

$$\sum_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T}} \dim J_{U,{}^x\psi_{Wh}}(V_0(\chi')) = d[\widetilde{T}^{(n)}\widetilde{T}_* : \widetilde{T}^{(n)}] = d[\widetilde{T}_* : \widetilde{T}_*^{(n)}]$$

Now let us return to the setup of the metaplectic tensor product. Let

$$\Theta_{\widetilde{M}}(\chi') \cong (\Theta(\chi'_1) \tilde{\otimes} \cdots \tilde{\otimes} \Theta(\chi'_k))_\omega$$

be an exceptional representation of $\widetilde{M}$. Let $d_i = \dim J_{U_{\mathrm{GL}_{r_i}}, \psi_{(r_i)}} \Theta(\chi'_i)$. Choose representatives for $\widetilde{T}_i^{(n)} \backslash \widetilde{T}_i$, and combine them together, we get a set of representatives of $\widetilde{T}^{(n)} \backslash \widetilde{T}$. Thus,

$$\sum_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T}} \dim J_{U_M, {}^x\psi_\lambda}(V_0(\chi'_1) \otimes \cdots \otimes V_0(\chi'_k)) = \prod_{i=1}^{k} d_i[\widetilde{T}_{*,i} : \widetilde{T}_{*,i}^{(n)}].$$

Proposition 3.6.1.

$$\dim J_{U_M,\psi_\lambda}(\Theta_{\widetilde{M}}(\chi')) = \frac{\prod_{i=1}^k d_i[\widetilde{T}_{*,i} : \widetilde{T}_{*,i}^{(n)}]}{[\widetilde{H} : \widetilde{M}^{(n)}]}.$$

Proof. Write $\pi^{(n)} = V_0(\chi_1') \otimes \cdots \otimes V_0(\chi_k')$. We have

$$J_{U_M,\psi_\lambda}(\mathrm{Ind}_{\widetilde{M}^{(n)}}^{\widetilde{M}} \pi^{(n)}) \cong \bigoplus_{x \in \widetilde{M}^{(n)} \backslash \widetilde{M}} J_{U_M, {}^x\psi_\lambda}(\pi^{(n)}) = \bigoplus_{x \in \widetilde{T}^{(n)} \backslash \widetilde{T}} J_{U_M, {}^x\psi_\lambda}(\pi^{(n)}).$$

By Lemma 3.4.5, the dimension of the left-hand side is $[\widetilde{H} : \widetilde{M}^{(n)}] \dim J_{U_M,\psi_\lambda}(\Theta_{\widetilde{M}}(\chi'))$. The dimension of the right-hand side is $\prod_{i=1}^k d_i[\widetilde{T}_{*,i} : \widetilde{T}_{*,i}^{(n)}]$. This proves the result. □

Now we proceed to simplify this formula. The above formula shows that $[\widetilde{H} : \widetilde{M}^{(n)}]$ is an invariant for $\Theta_{\widetilde{M}}(\chi')$. We calculate it by choosing good inducing data. Let π be an irreducible constituent of $\Theta_{\widetilde{M}}(\chi')|_{Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}}$. Recall

$$\Theta_{\widetilde{M}}(\chi')|_{Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}} = \bigoplus_{x \in Z_{\widetilde{\mathrm{GL}}_r} \widetilde{T}^{(n)} \backslash \widetilde{T} / \widetilde{T}_*} {}^x\pi = \bigoplus_{x \in \widetilde{T}^{(n)} \widetilde{T}_* \backslash \widetilde{T}} {}^x\pi.$$

Apply the induction functor $\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}}^{\widetilde{M}}$ and use Lemma 3.4.4 on the right-hand side. This gives

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}}^{\widetilde{M}} (\Theta_{\widetilde{M}}(\chi')|_{Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}}) = [\widetilde{T} : \widetilde{T}^{(n)} \widetilde{T}_*][\widetilde{H} : Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}] \Theta_{\widetilde{M}}(\chi').$$

Apply the Jacquet functor $J_{U_M}(-)$. This gives

$$\mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)} \cap \widetilde{T}}^{\widetilde{T}} J_{U_M}(\Theta_{\widetilde{M}}(\chi')|_{Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}}) = [\widetilde{T} : \widetilde{T}^{(n)} \widetilde{T}_*][\widetilde{H} : Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}] J_{U_M}(\Theta_{\widetilde{M}}(\chi')).$$

Comparing the dimensions and using $[\widetilde{T} : Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)} \cap \widetilde{T}] = [\widetilde{M} : Z_{\widetilde{\mathrm{GL}}_r} \widetilde{M}^{(n)}]$ gives

$$[\widetilde{M} : \widetilde{H}] = [\widetilde{T} : \widetilde{T}^{(n)} \widetilde{T}_*].$$

Thus

$$\begin{aligned} [\widetilde{H} : \widetilde{M}^{(n)}] &= \frac{[\widetilde{M} : \widetilde{M}^{(n)}]}{[\widetilde{T} : \widetilde{T}^{(n)} \widetilde{T}_*]} = \frac{[\widetilde{T} : \widetilde{T}^{(n)}]}{[\widetilde{T} : \widetilde{T}^{(n)} \widetilde{T}_*]} \\ &= [\widetilde{T}^{(n)} \widetilde{T}_* : \widetilde{T}^{(n)}] = [\widetilde{T}_* : \widetilde{T}_*^{(n)}]. \end{aligned}$$

Theorem 3.6.2.

$$\dim J_{U_M,\psi_\lambda}(\Theta_{\widetilde{M}}(\chi')) = \frac{\prod_{i=1}^k [\widetilde{T}_{*,i} : \widetilde{T}_{*,i}^{(n)}]}{[\widetilde{T}_* : \widetilde{T}_*^{(n)}]} \prod_{i=1}^k d_i.$$

Remark 3.6.3. We can see that the same calculation is true for the principal series representations.

Let us mention some immediate corollaries.

Corollary 3.6.4. *Suppose* $|n|_F = 1$. *If* $r_i > n$, *for some* i, *then*

$$J_{U_M,\psi_\lambda}(\Theta_{\widetilde{M}}(\chi')) = 0.$$

Proof. This is because when $r_i > n$, $d_i = 0$. □

Corollary 3.6.5. *Suppose* $|n|_F = 1$. *Let* $\Theta_r(\chi')$ *be an exceptional representation of* $\widetilde{\mathrm{GL}}_r$. *If* $r_i > n$ *for some* i, *then* $J_{U,\psi_\lambda}(\Theta_r(\chi')) = 0$. *In other words, there is no semi-Whittaker functional on* $\Theta_r(\chi')$.

Proof. In fact,

$$J_{U,\psi_\lambda}(\Theta_r(\chi')) = J_{U_M,\psi_\lambda}(J_R(\Theta_r(\chi'))) = J_{U_M,\psi_\lambda}(\Theta_{\widetilde{M}}({}^{w^M}\chi' \cdot \delta_P^{1/2})) = 0.$$

□

The following corollaries are true without $|n|_F = 1$.

Corollary 3.6.6. *When* $r_i \leq n$ *for all* i, $J_{U_M,\psi_\lambda}(\Theta_{\widetilde{M}}(\chi')) \neq 0$.

Corollary 3.6.7. *When* $r_i \leq n$ *for all* i, $J_{U,\psi_\lambda}(\Theta_r(\chi')) \neq 0$.

Now assume $|n|_F = 1$. Explicit constructions of maximal abelian groups may help us simplify the formula further. Indeed,

$$\begin{aligned}[][\widetilde{T}_* : \widetilde{T}_*^{(n)}] &= \frac{[\widetilde{T} : \widetilde{T}_*^{(n)}]}{[\widetilde{T} : \widetilde{T}_*]} = \frac{[\widetilde{T} : \widetilde{T}^{(n)}][\widetilde{T}^{(n)} : \widetilde{T}_*^{(n)}]}{[\widetilde{T} : \widetilde{T}_*^{\mathrm{st}}]} \\ &= \frac{[\widetilde{T} : \widetilde{T}^{(n)}][\widetilde{T}^{(n)} : \widetilde{T}_*^{(n)}][\widetilde{T}_*^{\mathrm{st}} : \widetilde{T}_{\mathfrak{o}}]}{[\widetilde{T} : \widetilde{T}_{\mathfrak{o}}]}.\end{aligned}$$

Notice that

$$\frac{\prod_{i=1}^k [\widetilde{T}_i : \widetilde{T}_i^{(n)}]}{[\widetilde{T} : \widetilde{T}^{(n)}]} = \frac{\prod_{i=1}^k [\widetilde{T}_i^{(n)} : \widetilde{T}_{*,i}^{(n)}]}{[\widetilde{T}^{(n)} : \widetilde{T}_*^{(n)}]} = \frac{\prod_{i=1}^k [\widetilde{T}_i : \widetilde{T}_{\mathfrak{o},i}]}{[\widetilde{T} : \widetilde{T}_{\mathfrak{o}}]} = 1.$$

Combining with Theorem 3.6.2, we obtain the following formula.

Theorem 3.6.8. *When* $|n|_F = 1$,

$$\dim J_{U_M,\psi_\lambda}(\Theta_{\widetilde{M}}(\chi')) = \frac{\prod_{i=1}^k [\widetilde{T}_{*,i}^{\mathrm{st}} : \widetilde{T}_{\mathfrak{o},i}]}{[\widetilde{T}_*^{\mathrm{st}} : \widetilde{T}_{\mathfrak{o}}]} \prod_{i=1}^k d_i.$$

When r is a multiple of n, we obtain the following uniqueness result.

Corollary 3.6.9. *Assume $|n|_F = 1$. If $r = mn$, and $\lambda = (n^m)$, then $J_{U,\psi_\lambda}(\Theta_r(\chi'))$ is one-dimensional.*

Proof. Under these assumptions, we have $\gcd(n, 2rc+r-1) = 1$. Therefore $Z_{\widetilde{\mathrm{GL}}_r} \subset \widetilde{T}_{\mathfrak{o}}$, and $[\widetilde{T}_*^{\mathrm{st}} : \widetilde{T}_{\mathfrak{o}}] = 1$. Similarly, $Z_{\widetilde{\mathrm{GL}}_n} \subset \widetilde{T}_{\mathfrak{o},i}$, and $[\widetilde{T}_{*,i}^{\mathrm{st}} : \widetilde{T}_{\mathfrak{o},i}] = 1$. By Proposition 3.1.3, $d_i = 1$ for all i. Therefore $\dim J_{U,\psi_\lambda}(\Theta_r(\chi')) = 1$. □

The uniqueness result also holds when $|n|_F \neq 1$. Indeed, let Ω be a maximal isotropic subgroup of the Hilbert symbol, then

$$\widetilde{T}_* := \{(\mathrm{diag}(t_1, \cdots, t_r), \zeta) : t_i \in \Omega\}$$

is a maximal abelian subgroup of $\widetilde{T}$, and $\widetilde{T}_*^{(n)} := Z_{\widetilde{M}^{(n)}} \cdot (\widetilde{T}_* \cap \widetilde{T}^{(n)})$ is a maximal abelian subgroup of $\widetilde{T}^{(n)}$. Notice $Z_{\widetilde{M}^{(n)}} = \widetilde{Z_{M^{(n)}}}$. Moreover, $[\widetilde{T}_* : \widetilde{T}_*^{(n)}] = \dfrac{[\widetilde{T} : \widetilde{T}_*]}{[\widetilde{T} : \widetilde{T}_*^{(n)}]}$ and

$$\frac{\prod_{i=1}^k [\widetilde{T}_i : \widetilde{T}_{*,i}]}{[\widetilde{T} : \widetilde{T}_*]} = \frac{\prod_{i=1}^k [\widetilde{T}_i : \widetilde{T}_{*,i}^{(n)}]}{[\widetilde{T} : \widetilde{T}_*^{(n)}]} = 1.$$

Combining this uniform description with Theorem 3.6.2, we are done.

Theorem 3.6.10. *Suppose $r = mn$, and M corresponds to the partition (n^m). In this case, $J_{U,\psi_\lambda}(\Theta_r(\chi'))$ is one-dimensional.*

Remark 3.6.11. Recall that the metaplectic cover $\widetilde{\mathrm{GL}}_r$ depends on an implicit choice of the modulus class $c \in \mathbb{Z}/n\mathbb{Z}$. Our results are true for all $c \in \mathbb{Z}/n\mathbb{Z}$. This is clear for the vanishing result (Corollary 3.6.5) and nonvanishing result (Corollary 3.6.7). For the uniqueness result, notice that when $r = mn$, $Z_{\widetilde{\mathrm{GL}}_r} = \{z^n I_r : z \in F^\times\}$. This fact is independent of c. Thus the proof of Corollary 3.6.9 is independent of c.

Remark 3.6.12. When $n = 2$, this is [BG92] Proposition 1.3 (i). Indeed, when $r = 2k$ and the partition is (2^k), this follows from Theorem 3.6.10. When $n = 2, r = 2k+1$, and M corresponds to the partition $(2^k 1)$. In this case, $d_i = 1$ for all i and $[\widetilde{T}_*^{\mathrm{st}} : \widetilde{T}_{\mathfrak{o}}] = [F^\times : F^{\times 2}\mathfrak{o}^\times]$. Moreover, $[\widetilde{T}_{*,i}^{\mathrm{st}} : \widetilde{T}_{\mathfrak{o},i}] = 1$ if $r_i = 2$; and $= [F^\times : F^{\times 2}\mathfrak{o}^\times]$ if $r_i = 1$. The twisted Jacquet module of $\Theta_{\widetilde{M}}(\chi')$ is again one-dimensional.

Remark 3.6.13. Assume $\Theta(\chi')$ has unique semi-Whittaker functional at almost all primes, a global argument similar to [KP84] Section II should show that $\Theta(\chi')$ has unique semi-Whittaker functional at all primes.

Chapter 4

Semi-Whittaker Coefficients: Global Theory

4.1 Theta representations

Let $n \geq 2$. Let F be a number field containing a full set of nth roots of unity μ_n, and let $\mathbb{A}$ denote the adeles of F. For $r \geq 2$, let $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ denote an n-fold cover of the general linear group.

We recall the definition of the global theta representations. These representations were constructed in [KP84] using the residues of Eisenstein series, as follows. Let B be the standard Borel subgroup of GL_r, and $T \subset B$ denote the maximal torus of GL_r. Let $\underline{s}$ be a multi-complex variable, and define the character $\mu_{\underline{s}}$ of $T(\mathbb{A})$ by $\mu_{\underline{s}}(\mathrm{diag}(a_1, \cdots, a_r)) = \prod_i |a_i|^{s_i}$. Let $Z(\widetilde{T}(\mathbb{A}))$ denote the center of $\widetilde{T}(\mathbb{A})$. Let $\omega_{\underline{s}}$ be a genuine character of $Z(\widetilde{T}(\mathbb{A}))$ such that $\omega_{\underline{s}} = \mu_{\underline{s}} \circ p$ on $\{(t^n, 1) | t \in T(\mathbb{A})\}$, where p is the canonical projection from $\widetilde{T}(\mathbb{A})$ to $T(\mathbb{A})$. Choose a maximal abelian subgroup A of $\widetilde{T}(\mathbb{A})$, extend this character to a character of A, and induce it to $\widetilde{T}(\mathbb{A})$. Then extend it trivially to $\widetilde{B}(\mathbb{A})$ using the canonical projection from $\widetilde{B}(\mathbb{A})$ to $\widetilde{T}(\mathbb{A})$, and further induce it to the group $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. We abuse the notation slightly and write this induced representation $\mathrm{Ind}_{\widetilde{B}(\mathbb{A})}^{\widetilde{\mathrm{GL}}_r(\mathbb{A})} \mu_{\underline{s}} \delta_B^{1/2}$. It follows from [KP84] that this construction is independent of the choice of A and of the extension of characters. Forming the Eisenstein series $E(\underline{s}, g)$ attached to this induced representation, it following from [KP84], that when $n(s_i - s_{i+1}) = 1$ for $1 \leq i \leq r-1$, this Eisenstein series has a nonzero residue representation. Let Λ be such a pole, and we write the residue representation as $\Theta_{r,\Lambda}$. The poles where we take the residues are usually clear in the context, thus sometimes we omit it from the notation. The global theta representation Θ_r is the restricted tensor product of the local exceptional representations $\Theta_{r,v}$. It is shown in [KP84] Section II that Θ_r is generic if and only if $r \le n$.

4.2 Vanishing results

Proposition 4.2.1. *Let θ be in the space of Θ_r. Let $\lambda = (r_1 \cdots r_k)$ be a partition of r. If there is an $r_i > n$ for some i, then*

$$\int\limits_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\ du \equiv 0$$

for all choices of data.

Proof. Let v be a place of F such that F_v is a non-archimedean local field where $|n|_v = 1$ and Θ_r is unramified. If

$$\int\limits_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\ du$$

is nonzero, then the functional

$$\theta \longmapsto \int\limits_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\ du$$

induces a nonzero functional on $\Theta_{r,v}$ which factors through the twisted Jacquet module for the character $\psi_\lambda(u)$ on the group $U(F_v)$. This contradicts the local result. □

4.3 Constant terms I

Let $\lambda = (r_1 \cdots r_k)$ be a partition of r. Let P_λ be the standard parabolic subgroup of GL_r with Levi subgroup M_λ and unipotent radical U_λ.

The goal for this section is to determine the constant term of Θ_r. We first compute the constant term of the Eisenstein series along U_λ. This turns out to be a sum of Eisenstein series on $\widetilde{M_\lambda}(\mathbb{A})$, over a subset of the Weyl group W. We exchange the constant term operator and the multi-residue operator, and the constant terms actually span a "theta representation" on $\widetilde{M}_\lambda(\mathbb{A})$. We then review the construction of the global metaplectic tensor product in Section 4.4, and show that the theta representation on $\widetilde{M}_\lambda(\mathbb{A})$ is actually the global metaplectic tensor product of Θ_{r_i}'s.

Proposition 4.3.1. *If $\theta \in \Theta_r$, then the constant term*

$$m \mapsto \int_{U_\lambda(F)\backslash U_\lambda(\mathbb{A})} \theta(um)\, du, \qquad m \in \widetilde{M}_\lambda(\mathbb{A})$$

is the residue of an Eisenstein series on $\widetilde{M}_\lambda(\mathbb{A})$.

If we vary $\theta \in \Theta_r$, then the constant terms of θ's span an irreducible automorphic representation of $\widetilde{M}_\lambda(\mathbb{A})$. We denote it by $\Theta_{\widetilde{M}_\lambda}$. As in the general linear case, $\Theta_{\widetilde{M}_\lambda}$ is the restricted tensor product of local theta representations of $\widetilde{M}_\lambda(F_v)$.

Proof. Let $\theta(g) = \mathrm{Res}_{\underline{s}=\Lambda} E(\phi, \underline{s}, g)$. We first compute the constant term of the Eisenstein series $E(\phi, \underline{s}, g)$ along P_λ. To do this, we introduce the set W_λ which consists of elements w^{-1} such that $w^{-1}(\beta) > 0$ for any Φ_λ^+, and $wTw^{-1} \subset M_\lambda$. By Moeglin and Waldspurger [MW95] Proposition 2.1.7(2),

$$\begin{aligned} E(\phi, \underline{s}, g)_{P_\lambda} &= \sum_{w^{-1}\in W_\lambda} \sum_{\gamma \in (wBw^{-1}\cap M_\lambda)(F)\backslash M_\lambda(F)} T(w, \underline{s})\phi(\underline{s})(\gamma g) \\ &= \sum_{w^{-1}\in W_\lambda} E^{\widetilde{M}_\lambda}(T(w, \underline{s})\phi(\underline{s}), w\underline{s}, g) \end{aligned}$$

Let Λ denote the pole of $E(\phi, \underline{s}, g)$ as in Section 4.1. To compute the constant term of theta function along P_λ, we use the fact that the multi-residue operator $\lim_{\underline{s}\to\Lambda} \prod_{i=1}^{r-1}(ns_i - ns_{i+1} - 1)$ and the constant term operator commute. Following an argument as in the proof of Offen and Sayag [OS08] Lemma 2.4, we deduce that after applying the multi-residue operator, the only term left is the one corresponding to w^{M_λ}.

We identify the set of simple roots with $\{(i, i+1) : 1 \le i \le r-1\}$. Given $w^{-1} \in W_\lambda$, let $\Delta^1(w) = \{i : \alpha = (i, i+1), w^{-1}(\alpha) < 0\}$. Notice that by the definition of W_λ, $\Delta^1(w)$ is contained in $\{r_1, r_1 + r_2, \cdots\}$. Then the normalized intertwining operator

$$N(w, \underline{s}) = \prod_{i\in\Delta^1(w)} (ns_i - ns_{i+1} - 1) T(w, \underline{s})$$

is holomorphic at Λ. Notice that the action of w on $\underline{s}$ is

$$w(s_1, \cdots, s_r) = (s_{w^{-1}(1)}, \cdots, s_{w^{-1}(r)}).$$

Let

$$\Delta^2(w) = \{i : w^{-1}(i+1) - w^{-1}(i) = 1\} \backslash \{r_1, r_1 + r_2, \cdots\}.$$

Then the normalized Eisenstein series

$$\prod_{i \in \Delta^2(w)} (ns_i - ns_{i+1} - 1) E^{\widetilde{M}_\lambda}(N(w, \underline{s})\phi(\underline{s}), w\underline{s}, g)$$

is holomorphic at Λ. Thus, the terms corresponding to w^{-1} survives after taking multi-residue if and only if

$$\Delta^1(w) \cup \Delta^2(w) = \{1, \cdots, r-1\}.$$

This implies that $\Delta^1(w) = \{r_1, r_1 + r_2, \cdots\}$ and w permutes blocks of M_λ. The only possibility is $w = w^{M_\lambda}$. Thus we have shown the following identity

$$\theta(g)_{P_\lambda} = \mathrm{Res}_{\underline{s}=\Lambda} E^{\widetilde{M}_\lambda}(T(w^{M_\lambda}, \underline{s})\phi(\underline{s}), w^{M_\lambda}\underline{s}, g).$$

This finishes the proof.

□

4.4 Global metaplectic tensor product

The global metaplectic tensor product was first given in [Tak16] Section 5, and a simplified version is given in [Takar]. We briefly review the latter construction here.

Assume (π, V_π) is an automorphic representation of G, and V_π is a space of functions or maps on the group G, and π is the representation of G on V_π defined by right translation. Let $H \subset G$ be a subgroup. Then we define $\pi||_H$ to be the representation of H realized in the space

$$V_{\pi||_H} := \{f|_H : f \in V_\pi\}$$

of restrictions of $f \in V_\pi$ to H, on which H acts by right translation.

Let π_i be a genuine irreducible automorphic unitary representation of $\widetilde{\mathrm{GL}}_{r_i}(\mathbb{A})$. Let $H_i = \mathrm{GL}_{r_i}(F)\widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A})$, and $\sigma_i = \pi_i||_{H_i}$. Then the restriction $\pi_i|_{H_i}$ is completely reducible ([Takar], Proposition 3.6). Hence σ_i is a subrepresentation of $\pi_i|_{H_i}$.

Note that H_i is indeed a closed subgroup of $\widetilde{\mathrm{GL}}_{r_i}(\mathbb{A})$. By the product formula for the Hilbert symbol and block-compatibility of the cocycle, we have the natural surjection

$$H_1 \times \cdots \times H_k \to M(F)\widetilde{M}^{(n)}(\mathbb{A}).$$

Then consider the space

$$V_{\sigma_1} \otimes \cdots \otimes V_{\sigma_k}$$

as functions on direct product $H_1 \times \cdots \times H_k$, which gives rise to a representation of $H_1 \times \cdots \times H_k$. If $\varphi_i \in V_{\sigma_i}$ for $i = 1, \cdots, k$, we denote this function by

$$\varphi_1 \otimes \cdots \otimes \varphi_k,$$

and denote the space generated by those function by V_σ. These functions can be viewed as "automorphic forms" on $M(F)\widetilde{M}^{(n)}(\mathbb{A})$. The group $M(F)\widetilde{M}^{(n)}(\mathbb{A})$ acts on V_σ by right translation. Denote this representation by σ. This representation is completely reducible ([Takar] Proposition 3.8).

Fix an irreducible subrepresentation τ of σ. Then the abelian group $Z_{\widetilde{\mathrm{GL}}_r(\mathbb{A}) \cap M(F)\widetilde{M}^{(n)}(\mathbb{A})}$ acts a a character ω_τ ([Takar] Lemma 3.12). Choose a "Hecke character" ω on $Z_{\widetilde{\mathrm{GL}}_r(\mathbb{A})}$ by extending ω_τ. Then one can extend τ to a representation τ_ω on $Z_{\widetilde{\mathrm{GL}}_r(\mathbb{A})} M(F)\widetilde{M}^{(n)}(\mathbb{A})$. Consider the smooth induced representaion

$$\Pi(\tau_\omega) := \mathrm{Ind}_{Z_{\widetilde{\mathrm{GL}}_r(\mathbb{A})} M(F)\widetilde{M}^{(n)}(\mathbb{A})}^{\widetilde{M}(\mathbb{A})} \tau_\omega.$$

We can view $\Pi(\tau_\omega)$ as a subrepresentation of $\mathcal{A}(\widetilde{M})$, which is the space of automorphic forms on $\widetilde{M}(\mathbb{A})$. Moreover, $\Pi(\tau_\omega)$ has an irreducible subrepresentation ([Takar] Proposition 3.15). Choose such a representation and denote it by π_ω. Then we call π_ω a metaplectic tensor product of $\pi_1, \cdots, \pi_k$ with respect to the character ω and write

$$\pi_\omega = (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega.$$

The representation π_ω has the desired local-global compatibility. Moreover, it is unique up to equivalence, and depends only on $\pi_1, \cdots, \pi_k$ and ω ([Takar] Theorem 3.18).

4.5 Constant term II

We give the second description of the constant term of the theta function. We show that the theta representation on $\widetilde{M}_\lambda(\mathbb{A})$ is in fact the global metaplectic tensor product of theta representations on $\widetilde{\mathrm{GL}}_{r_i}(\mathbb{A})$.

Theorem 4.5.1. *If $\theta \in \Theta_r$, the constant term*

$$m \mapsto \int_{U_\lambda(F)\backslash U_\lambda(\mathbb{A})} \theta(um)\, du, \qquad m \in \widetilde{M}_\lambda(\mathbb{A})$$

is in the space $\Theta_{r_1}\tilde{\otimes}\cdots\tilde{\otimes}\Theta_{r_k}$. Indeed,

$$\Theta_{\widetilde{M}_\lambda} \cong \Theta_{r_1}\tilde{\otimes}\cdots\tilde{\otimes}\Theta_{r_k}.$$

Here, the global metaplectic tensor product is with respect to the central character ω of $\Theta_{\widetilde{M}_\lambda}$. The poles that we use to define Θ_{r_i} are specified in the proof.

Proof. Write $\sigma_i = \Theta_{r_i}||_{H_i}$ for $i = 1, \cdots, k$. As explained above, the representation $\sigma_1\tilde{\otimes}\cdots\tilde{\otimes}\sigma_k$ descents to a representation σ on $M(F)\widetilde{M}^{(n)}(\mathbb{A})$. It suffices to show that

$$\Theta_{M_\lambda}||_{M(F)\widetilde{M}^{(n)}(\mathbb{A})} \hookrightarrow \sigma.$$

Notice the space σ contains the metaplectic tensor products with respect to all possible characters ω.

Before we prove this claim we would like to introduce some notations. Let $E(\underline{s}, g)$ be the Eisenstein series of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ and let Λ be the pole to define the theta function. Let Λ_P be defined such that μ_{Λ_P} is the modular quasicharacter of GL_k with respect to P_λ. Write $\Lambda = (\Lambda_k, \cdots, \Lambda_1)$, where Λ_i is of size r_i. Write $\Lambda_P = (\Lambda_{P,1}, \cdots, \Lambda_{P,k})$ such that $\Lambda_{P,i}$ is of size r_i. Notice that all the entries in $\Lambda_{P,i}$ are the same.

Let $f \in \Theta_{M_\lambda}||_{M(F)\widetilde{M}^{(n)}(\mathbb{A})}$. This means that f is the restriction of the residue of an Eisenstein series $E^{\widetilde{M}_\lambda}(\underline{s}, g)$ to $M(F)\widetilde{M}^{(n)}(\mathbb{A})$. Indeed, if $g \in M(F)\widetilde{M}^{(n)}(\mathbb{A})$, then

$$\begin{aligned} f(g) =& \mathrm{Res}_{\underline{s}=w^M(\Lambda)+\Lambda_P} E(\underline{s}, g) \\ =& \mathrm{Res}_{\underline{s}=w^M(\Lambda)+\Lambda_P} \sum_{\gamma \in B_M(F)\backslash M(F)} \phi(\underline{s})(\gamma g) \\ =& \mathrm{Res}_{\underline{s}=w^M(\Lambda)+\Lambda_P} \sum_{\gamma \in B_M^{(n)}(F)\backslash M^{(n)}(F)} \phi(\underline{s})(\gamma g). \end{aligned}$$

The last equality follows from the following fact: $M^{(n)}(F) \hookrightarrow M(F)$ induces a bijection $B_M^{(n)}(F)\backslash M^{(n)}(F) \leftrightarrow B(F)\backslash M(F)$.

Without loss of generality, we may assume that $\phi \in \mathrm{Ind}_{\widetilde{B}^{(n)}(\mathbb{A})}^{\widetilde{M}^{(n)}(\mathbb{A})} \mu_{\underline{s}}\delta_M^{1/2} \subset \mathrm{Ind}_{\widetilde{B}_M(\mathbb{A})}^{\widetilde{M}(\mathbb{A})} \mu_{\underline{s}}\delta_M^{1/2}$ and furthermore it is decomposable: $\phi = \phi_1 \tilde{\otimes} \cdots \tilde{\otimes} \phi_k$, where $\phi_i \in \mathrm{Ind}_{\widetilde{B}_i^{(n)}(\mathbb{A})}^{\widetilde{\mathrm{GL}}_{r_i}^{(n)}(\mathbb{A})} \mu_{\underline{s}_i}\delta_{B_i}^{1/2}$. Write $g = \mathrm{diag}(g_1, \cdots, g_k)$ and $\gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_k)$. Then we can naturally view $f = f_1 \tilde{\otimes} \cdots \tilde{\otimes} f_k$, where

$$f_i(g_i) = \mathrm{Res}_{\underline{s}_i = \Lambda_i + \Lambda_{P,i}} \sum_{\gamma_i \in B_i^{(n)}\backslash \mathrm{GL}_{r_i}^{(n)}(F)} \phi_{\underline{s}_i}(\gamma_i g_i).$$

This means that $f_i \in \Theta_{r_i, \Lambda_i + \Lambda_{P,i}}$. We are done.

$\square$

4.6 Global nonvanishing

Now we prove the global nonvanishing results. Let λ be a partition of r. Define the Levi subgroup $M = M_\lambda$ as usual. Define the semi-Whittaker functional ψ_λ as in the local case. We also write $\psi_\lambda(u) = \psi_1(u_1)\cdots\psi_k(u_k)$ if $u = \mathrm{diag}(u_1, \cdots, u_k) \in U \cap M$.

Theorem 4.6.1. *If $r_i \leq n$ for all i, then*

$$\int_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\, du$$

is nonzero for some choices of $\theta \in \Theta_r$ and $g \in \widetilde{\mathrm{GL}}_r(\mathbb{A})$.

Proof. Notice that

$$\int_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\, du = \int_{U_M(F)\backslash U_M(\mathbb{A})} \int_{U_\lambda(F)\backslash U_\lambda(\mathbb{A})} \theta(vug)\, dv\, \psi_\lambda(u)\, du.$$

By Proposition 4.3.1, it suffices to show that

$$\int_{U_M(F)\backslash U_M(\mathbb{A})} f(ug)\psi_\lambda(u)\, du \neq 0$$

for some choices of $\theta \in \Theta_{\widetilde{M}}$ and $g \in \widetilde{M}(\mathbb{A})$. We now use notations in the proof of Theorem 4.5.1. Notice that the character ω in Theorem 4.5.1 does not contribute anything in this integral. Thus it suffices to show that

$$\int_{U_M(F)\backslash U_M(\mathbb{A})} f(u)(1)\psi_\lambda(u)\ du \neq 0$$

for some $f \in \mathrm{Ind}_{M(F)\widetilde{M}^{(n)}(\mathbb{A})}^{\widetilde{M}(\mathbb{A})}\sigma$. Here $f(u)$ is in σ and we use $f(u)(1)$ to denote its value at 1. Notice that $U_M(\mathbb{A}) \subset M(F)\widetilde{M}^{(n)}(\mathbb{A})$. Thus $f(u)(1) = f(1)(u)$.

Without of loss of generality, we can choose f such that $f(1)$ is a simple tensor $f_1\tilde{\otimes}\cdots\tilde{\otimes}f_k$, where $f_i \in \sigma_i$. Moveover, we can choose f_i such that the Whittaker coefficient of f_i is nonzero, i.e. $\int_{U_{\mathrm{GL}_{r_i}}(F)\backslash U_{\mathrm{GL}_{r_i}}(\mathbb{A})} f_i(u_i g_i)\psi_i(u_i)\ du_i \neq 0$ for all i. (This is because when $r_i \leq n$, Θ_{r_i} is generic.)

Thus,

$$\begin{aligned}
&\int_{U_M(F)\backslash U_M(\mathbb{A})} f(u)(1)\psi_\lambda(u)\ du \\
=&\int_{U_M(F)\backslash U_M(\mathbb{A})} f(1)(u)\psi_\lambda(u)\ du \\
=&\prod_{i=1}^{k}\int_{U_{\mathrm{GL}_{r_i}}(F)\backslash U_{\mathrm{GL}_{r_i}}(\mathbb{A})} f_i(u_i)\psi_i(u_i)\ du_i \neq 0.
\end{aligned}$$

This proves the theorem. $\square$

Chapter 5

Unipotent Orbits and Fourier Coefficients

For the rest of this paper, we turn to the Fourier coefficients associated with general unipotent orbits. In this section, we explain how to associate a set of Fourier coefficients with a unipotent orbit. General references for unipotent orbits are Carter [Car93] and Collingwood-McGovern [CM93]. For the local version of this association see [Mœg96, MW87]. For global details see Jiang-Liu [JL13] and Ginzburg [Gin06, Gin14]. The associated Fourier coefficients are described as integration over certain unipotent subgroups, and the metaplectic cocycles do not contribute any nontrivial factors. To simplify notations, we only describe this association in the non-metaplectic setup.

We work with the global setup. Let F be a number field, and $\mathbb{A}$ be its adele ring. Fix a nontrivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. The unipotent orbits of GL_r are parameterized by partitions of r. Let $\mathcal{O} = (p_1 \cdots p_k)$ with $p_1 + \cdots + p_k = r$ be a unipotent orbit. We shall always assume $p_1 \geq p_2 \geq \cdots \geq p_k > 0$. To each p_i we associate the diagonal matrix

$$\mathrm{diag}(t^{p_i-1}, t^{p_i-3}, \cdots, t^{3-p_i}, t^{1-p_i}).$$

Combining all such diagonal matrices and arranging them in decreasing order of the powers, we obtain a one-dimensional torus $h_{\mathcal{O}}(t)$. For example, if $\mathcal{O} = (3^2 1)$, then

$$h_{\mathcal{O}}(t) = \mathrm{diag}(t^2, t^2, 1, 1, 1, t^{-2}, t^{-2}).$$

The one-dimensional torus $h_{\mathcal{O}}(t)$ acts on U by conjugation. Let α be a positive root and $x_\alpha(a)$ be the one-dimensional unipotent subgroup in U corresponding to the root α. There is a nonnegative integer m such that

$$h_{\mathcal{O}}(t)x_\alpha(a)h_{\mathcal{O}}(t)^{-1} = x_\alpha(t^m a). \tag{5.1}$$

On the subgroups $x_\alpha(a)$ which correspond to negative roots α, the torus $h_{\mathcal{O}}(t)$ acts with non-positive powers.

Given a nonnegative integer l, we denote by $U_l(\mathcal{O})$ the subgroup of U generated by all $x_\alpha(a)$ satisfying the Eq. (5.1) with $m \geq l$. We are mainly interested in $U_l(\mathcal{O})$ where $l = 1$ or $l = 2$.

Let

$$M(\mathcal{O}) = T \cdot \langle x_{\pm\alpha}(a) : h_{\mathcal{O}}(t)x_\alpha(a)h_{\mathcal{O}}(t)^{-1} = x_\alpha(a)\rangle.$$

The group $M(\mathcal{O})$ acts by conjugation on the abelian group $U_2(\mathcal{O})/U_3(\mathcal{O})$. If the ground field is algebraically closed, then under this action of $M(\mathcal{O})$ on the group $U_2(\mathcal{O})/U_3(\mathcal{O})$, there is an open orbit. Denote a representative of this orbit by u_2. It follows from the general theory that the connected component of the stabilizer of this orbit inside $M(\mathcal{O})$ is a reductive group. Denote by $Stab^0_{\mathcal{O}}$ this connected component of the stabilizer of u_2.

The group $M(\mathcal{O})(F)$ acts on the group of all characters of $U_2(\mathcal{O})(F)\backslash U_2(\mathcal{O})(\mathbb{A})$. Consider the subset of all characters such that over the algebraic closure, the connected component of the stabilizer inside $M(\mathcal{O})(F)$ is equal to $Stab^0_{\mathcal{O}}$. We denote such a character by $\psi_{U_2(\mathcal{O})}$. Given an automorphic function $\varphi(g)$ on $\mathrm{GL}_r(\mathbb{A})$ or its cover, the Fourier coefficient we want to consider is

$$\int\limits_{U_2(\mathcal{O})(F)\backslash U_2(\mathcal{O})(\mathbb{A})} \varphi(ug)\psi_{U_2(\mathcal{O})}(u)\, du.$$

In this way, we associate with each unipotent orbit $\mathcal{O}$ a set of Fourier coefficients. When the partition is $\mathcal{O} = (r)$, the Fourier coefficients associated to $\mathcal{O}$ are the Whittaker coefficients.

In order to perform root exchange as in Sections 5.1.1 and 5.2.1 below, we also work with a slightly different torus. Let

$$h'_{\mathcal{O}}(t) = \mathrm{diag}(t^{p_1-1}, \cdots, t^{1-p_1}, t^{p_2-1}, \cdots, t^{1-p_2}),$$

where after the first block of size p_1, the exponents of t are of non-increasing order. These two one-dimensional tori are conjugate by an element in the Weyl group of GL_r. Let $V_2(\mathcal{O})$, $\psi_{V_2(\mathcal{O})}$ be the corresponding unipotent subgroup and character, respectively.

Let us recall the partial ordering defined on the set of unipotent orbits. Given $\mathcal{O}_1 = (p_1 \cdots p_k)$ and $\mathcal{O}_2 = (q_1 \cdots q_l)$, we say that $\mathcal{O}_1 \geq \mathcal{O}_2$ if $p_1 + \cdots + p_i \geq q_1 + \cdots + q_i$ for all $1 \leq i \leq l$. If $\mathcal{O}_1$ is not greater than $\mathcal{O}_2$ and $\mathcal{O}_2$ is not greater than $\mathcal{O}_1$, we say that $\mathcal{O}_1$ and $\mathcal{O}_2$ are not comparable.

Definition 5.0.2. Let π be an automorphic representation of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. Let $\mathcal{O}(\pi)$ denote the set of unipotent orbits of GL_r defined as follows. A unipotent orbit $\mathcal{O} \in \mathcal{O}(\pi)$ if π has a nonzero Fourier coefficient which is associated with the unipotent orbit $\mathcal{O}$, and for all $\mathcal{O}' > \mathcal{O}$, π has no nonzero Fourier coefficient associated with $\mathcal{O}'$.

We already describe this association in the global setup. The corresponding local picture could be described analogously. We omit the details.

5.1 Unipotent Orbits: Local Results

We return to the local setup in this section. Fix positive integers n, r such that $|n|_F = 1$. Write $r = an + b$, where $a \in \mathbb{Z}_{\geq 0}$ and $0 \leq b < n$. Let $\Theta = \Theta_r$ be an exceptional representation on $\widetilde{\mathrm{GL}}_r$. The unipotent orbit attached to Θ is determined in this section. The key ingredients are the results on the semi-Whittaker functionals. We follow closely the approach given in Jiang-Liu [JL13], where they determine the unipotent orbits attached to the residual spectrum of the general linear groups. Here we give a local version with necessary modifications.

Theorem 5.1.1. *Let $\mathcal{O} = (p_1 \cdots p_k)$ be a unipotent orbit of GL_r.*

(1) *If $p_1 > n$, then $J_{U_2(\mathcal{O}),\psi_{U_2(\mathcal{O})}}(\Theta) = 0$ (or equivalently, $J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta) = 0$).*

(2) *If $\mathcal{O} = (n^a b)$, then $J_{U_2(\mathcal{O}),\psi_{U_2(\mathcal{O})}}(\Theta) \neq 0$ (or equivalently, $J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta) \neq 0$).*

Notice that any unipotent orbit greater than or not comparable with $(n^a b)$ must have $p_1 > a$. Thus we obtain the following result.

Theorem 5.1.2. *Let Θ be an exceptional representation of $\widetilde{\mathrm{GL}}_r$. Then*

$$\mathcal{O}(\Theta) = (n^a b).$$

The rest of this section is devoted to proving Theorem 5.1.1. This theorem is also proved in an unpublished work of Gordan Savin by using the Iwahori-Hecke algebras.

5.1.1 A general lemma

We start with a general lemma, which is used repeatedly in this section.

Let G be the rational points of a split algebraic group or a cover of such. Let $\mathfrak{u}$ be a maximal nilpotent Lie subalgebra of $\mathrm{Lie}(G)$. Let $\mathfrak{A}, \mathfrak{C}, \mathfrak{X}$ and $\mathfrak{Y}$ be Lie subalgebras of $\mathfrak{u}$, and let A, C, X, Y be the corresponding unipotent subgroups of G. Let ψ_C be a nontrivial character of C. We make the following assumptions:

(a) $C, X, Y \subset A$.

(b) X and Y are abelian, normalize C and preserve ψ_C.

(c) The commutators $x^{-1}y^{-1}xy$ lie in C, for all $x \in X, y \in Y$. In particular, Y normalizes $D = CX$ and X normalizes $B = CY$.

(d) $A = D \rtimes Y = B \rtimes X$.

(e) The set

$$\{x \mapsto \psi_C(x^{-1}y^{-1}xy) | y \in Y\}$$

is the group of all characters of X. Moreover, writing $x = \exp E, y = \exp S$, for $E \in \mathfrak{X}, S \in \mathfrak{Y}$, we have

$$\psi_C(xyx^{-1}y^{-1}) = \psi((E, S))$$

where $(\ ,\ )$ is a nondegenerate, bilinear pairing between $\mathfrak{X}$ and $\mathfrak{Y}$.

$$\begin{tikzcd}[column sep=small]
 & BX = A = DY \arrow[dl, dash, "X"'] \arrow[dr, dash, "Y"] & \\
B = CY \arrow[dr, dash, "Y"'] & & D = CX \arrow[dl, dash, "X"] \\
 & C &
\end{tikzcd}$$

Lemma 5.1.3. *Assume (a)-(e). Let π be a smooth representation of A. Extend ψ_C trivially to characters ψ_B of B and ψ_D of D. Then we have an isomorphism of C-modules*

$$J_{B,\psi_B}(\pi) \cong J_{D,\psi_D}(\pi).$$

Moreover,

$$J_{C,\psi_C}(\pi) = 0 \Longleftrightarrow J_{D,\psi_D}(\pi) = 0 \Longleftrightarrow J_{B,\psi_B}(\pi) = 0.$$

Proof. The first isomorphism is proved in Ginzburg-Rallis-Soudry [GRS99] Section 2.2. For the second statement, clearly if $J_{C,\psi_C}(\pi) = 0$, then

$$J_{D,\psi_D}(\pi) = J_X(J_{C,\psi_C}(\pi)) = 0.$$

Conversely, suppose $J_{D,\psi_D}(\pi) = 0$ (the other case can be treated similarly). There is a natural map

$$T : J_{C,\psi_C}(\pi) \to J_{D,\psi_D}(\pi) = 0$$

over D. This induces a map of A-modules

$$i : J_{C,\psi_C}(\pi) \to \mathrm{Ind}_D^A(J_{D,\psi_D}(\pi)) = 0.$$

It is shown in [GRS99] that i is injective. Thus $J_{C,\psi_C}(\pi) = 0$.

□

When X and Y are root subgroups, the above lemma is the local version of the root exchange in Friedberg-Ginzburg [FGar] Section 2.2 and Ginzburg [Gin15] Section 2.2.2. This is always the case in our application. The above assumptions can always be verified by the Steinberg relations.

5.1.2 Root exchange

Given a unipotent orbit $\mathcal{O} = (p_1 \cdots p_k)$, we define several unipotent subgroups of U. Let $U_{\mathcal{O}}$ be the subgroup of U consisting elements of the form

$$u = \begin{pmatrix} u_1 & n_1 \\ & u_2 \end{pmatrix},$$

where $u_1 \in \mathrm{GL}_{p_1}$ is unipotent, $n_1 \in \mathrm{Mat}_{p_1 \times (n-p_1)}$ with the last row being zero, and $u_2 \in U_2((p_2 \cdots p_k)) \subset \mathrm{GL}_{r-p_1}$. We define a character $\psi_{U_{\mathcal{O}}} : U_{\mathcal{O}} \to \mathbb{C}^\times$ as the product of the Whittaker character on u_1 and $\psi_{U_2((p_2\cdots p_k))}$ on u_2. We also define a unipotent subgroup $U'_{\mathcal{O}}$ of $U_{\mathcal{O}}$ by removing all the root subgroups U_α in the n_1 part, such that

$$h'_{\mathcal{O}}(t)x_\alpha(a)h'_{\mathcal{O}}(t)^{-1} = x_\alpha(ta). \tag{5.2}$$

The character $\psi_{U'_{\mathcal{O}}}$ is defined analogously as $\psi_{U_{\mathcal{O}}}$.

Remark 5.1.4. If p_i's have the same parity, then $U_{\mathcal{O}} = U'_{\mathcal{O}}$.

Lemma 5.1.5. *Let π be a smooth representation of $\widetilde{\mathrm{GL}}_r$.*

(1)

$$J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\pi) \cong J_{U'_{\mathcal{O}},\psi_{U'_{\mathcal{O}}}}(\pi).$$

(2)

$$J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\pi) = 0$$

if and only if

$$J_{U_{\mathcal{O}},\psi_{U_{\mathcal{O}}}}(\pi) = 0.$$

(3) *If p_i's have the same parity, then*

$$J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\pi) \cong J_{U_{\mathcal{O}},\psi_{U_{\mathcal{O}}}}(\pi).$$

Proof. Part (3) is clear from part (1) and Remark 5.1.4. We first prove part (1). The strategy is to use the root exchange lemma. Notice that any element of $V_2(\mathcal{O})$ has the following form:

$$u = \begin{pmatrix} u_1 & q \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} I_{p_1} & 0 \\ p & I_{n-p_1} \end{pmatrix},$$

where $u_1 \in \mathrm{GL}_{p_1}$ and $u_2 \in U_{(p_2\cdots p_k)} \subset \mathrm{GL}_{n-p_1}$ are unipotent matrices, and $p \in \mathrm{Mat}_{p_1\times(n-p_1)}$ and $q \in \mathrm{Mat}_{(n-p_1)\times p_1}$ are certain matrices to be described later. The character $\psi_{V_2(\mathcal{O})}$ is the product of Whittaker character on u_1 and $\psi_{U_2((p_2\cdots p_k))}$. We use the simple roots in u_1 to move root subgroups contained p to q. The desired twisted Jacquet module is obtained after we finish this process.

Let us give more details in the case $\mathcal{O} = (p_1p_2)$. The general case follows by the same argument. There are two cases to consider, depending on the parity of $p_1 - p_2$.

Case 1: $p_1 - p_2$ is even. Notice that in this case part (1) implies part (2) immediately.

We can write $u \in V_2(\mathcal{O})$ as

$$u = \begin{pmatrix} u_1 & n_1 \\ n_2 & u_2 \end{pmatrix}.$$

Here $u_1 \in \mathrm{GL}_{p_1}, u_2 \in \mathrm{GL}_{p_2}$ are unipotent matrices, and

$$n_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \in \mathrm{Mat}_{p_1 \times p_2},$$

where

$$a_1 \in \mathrm{Mat}_{(\frac{p_1-p_2}{2}) \times p_2}, b_1 \in \mathrm{Mat}_{p_2 \times p_2} \text{ is nilpotent}, c_1 = 0 \in \mathrm{Mat}_{(\frac{p_1-p_2}{2}) \times p_2},$$

and

$$n_2 = \begin{pmatrix} 0 & b_2 & c_2 \end{pmatrix} \in \mathrm{Mat}_{p_2 \times p_1}$$

where

$$0 \in \mathrm{Mat}_{p_2 \times (\frac{p_1-p_2}{2}+1)}, b_2 \in \mathrm{Mat}_{p_2 \times p_2} \text{ is upper triangular}, c_2 \in \mathrm{Mat}_{p_2 \times (\frac{p_1-p_2}{2}-1)}.$$

Now we apply the root exchange lemma. For the first column of b_2, the only nonzero entry corresponds to the root subgroup associated to the (negative) root $(p_1+1, \frac{p_1-p_2}{2}+2)$. We use the simple root $(\frac{p_1-p_2}{2}+1, \frac{p_1-p_2}{2}+2)$ in u_1. This replaces the root $(p_1+1, \frac{p_1-p_2}{2}+2)$ by the (positive) root $(\frac{p_1-p_2}{2}+1, p_1+1)$ in the integration region. Notice that $(\frac{p_1-p_2}{2}+1, p_1+1)$ is exactly the only missing entry in the first row of b_1.

Similarly, the i-th column of b_2 has i nonzero entries, corresponding to the roots

$$\left(j, \frac{p_1-p_2}{2}+i+1\right), \qquad j = p_1+1, \cdots, p_1+i.$$

We use the simple root $(\frac{p_1-p_2}{2}+i, \frac{p_1-p_2}{2}+i+1)$. By root exchange, these roots are moved to

$$\left(\frac{p_1-p_2}{2}+i, j\right), \qquad j = p_1+1, \cdots, p_1+i.$$

These are exactly the missing entries in the i-th row in b_1. The c_2 part can be handled similarly. Indeed, using the simple roots in u_1, entries in c_2 are moved to the first $(\frac{p_1-p_2}{2}-1)$ rows of c_1. Thus, in this case, we have shown that

$$J_{V_2(\mathcal{O}), \psi_{V_2(\mathcal{O})}}(\pi) \cong J_{U'_{\mathcal{O}}, \psi_{U'_{\mathcal{O}}}}(\pi).$$

Case 2: $p_1 - p_2$ is odd. The proof of part (1) is the same as case 1, with minor differences. Indeed, $u \in V_2(\mathcal{O})$ can be written as

$$u = \begin{pmatrix} u_1 & n_1 \\ n_2 & u_2 \end{pmatrix}.$$

Here $u_1 \in \mathrm{GL}_{p_1}, u_2 \in \mathrm{GL}_{p_2}$ are unipotent matrices, and

$$n_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \in \mathrm{Mat}_{p_1 \times p_2},$$

where

$$a_1 \in \mathrm{Mat}_{(\frac{p_1-p_2-1}{2}) \times p_2}, b_1 \in \mathrm{Mat}_{p_2 \times p_2} \text{ is nilpotent}, c_1 = 0 \in \mathrm{Mat}_{(\frac{p_1-p_2+1}{2}) \times p_2},$$

and

$$u_2 = \begin{pmatrix} 0 & b_2 & c_2 \end{pmatrix} \in \mathrm{Mat}_{p_2 \times p_1}$$

where

$$0 \in \mathrm{Mat}_{p_2 \times (\frac{p_1-p_2+1}{2})}, b_2 \in \mathrm{Mat}_{p_2 \times p_2} \text{ is nilpotent}, c_2 \in \mathrm{Mat}_{p_2 \times (\frac{p_1-p_2-1}{2})}.$$

There is no element in the first column of b_2, and the first entry of b_1 is missing. For the second column of b_2, the only nontrivial entry corresponds to the root $(p_1 + 1, \frac{p_1-p_2+1}{2} + 2)$. We use the simple root $(\frac{p_1-p_2+1}{2} + 1, \frac{p_1-p_2+1}{2} + 2)$ to move it to the positive root $(\frac{p_1-p_2+1}{2} + 1, p_1 + 1)$. This is the first entry of the second row of b_1. Now we only miss the second entry in the second row of b_1.

Similarly, the $(i+1)$-th column of b_2 has i entries, corresponding to the roots

$$(j, \frac{p_1 - p_2 + 1}{2} + i + 1), \qquad j = p_1 + 1, \cdots, p_1 + i.$$

We use the simple root $(\frac{p_1-p_2}{2} + i, \frac{p_1-p_2+1}{2} + i + 1)$. By root exchange, these roots are moved to

$$(\frac{p_1 - p_2 + 1}{2} + i, j), \qquad j = p_1 + 1, \cdots, p_1 + i.$$

Thus, after this process, we only miss the $(i+1)$-th entry in the $(i+1)$-th row in b_1. The c_2 part can be handled similarly, and the entries in c_2 are moved to the first $(\frac{p_1-p_2+1}{2} - 1)$ rows of c_1. The missing entries in b_1 are the diagonal entries, which are exactly the root subgroups that are removed in the definition of $U'_{\mathcal{O}}$; see Eq. (5.2). This finishes the proof of part (1).

For part (2), let Y be the subgroup of $V_2(\mathcal{O})$ such that $u_1 = I, u_2 = I$, $n_2, a_1, c_1 = 0$, and b_1 is diagonal. Then we can verify that Y normalize $U'_{\mathcal{O}}$ and preserve $\psi_{U_{\mathcal{O}}}$. Moreover, $U'_{\mathcal{O}}Y = U_{\mathcal{O}}$. By Lemma 5.1.3,

$$J_{U_{\mathcal{O}},\psi_{\mathcal{O}}}(\pi) = 0$$

if and only if

$$J_{U'_{\mathcal{O}},\psi'_{\mathcal{O}}}(\pi) = 0$$

if and only if

$$J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\pi) = 0.$$

For the general case, we need to proceed inductively. Notice that if we perform root exchange on the p_3 part using p_1, what is done in the previous steps are unchanged. Therefore, the lemma is true for a general unipotent orbit $\mathcal{O}$. □

5.1.3 Vanishing results

Now we prove the vanishing property of the twisted Jacquet modules of Θ attached to the unipotent orbits either greater than or not comparable with $(n^a b)$.

Let $V_{1^{m-1},r-m+1}$ be the unipotent radical of the parabolic subgroup $P_{1^{m-1},r-m+1}$ with Levi part $\mathrm{GL}_1^{\times(m-1)} \times \mathrm{GL}_{r-m+1}$. Let

$$\psi_{m-1}(v) = \psi(v_{1,2} + \cdots + v_{m-1,m}),$$

and

$$\tilde{\psi}_{m-1}(v) = \psi(v_{1,2} + \cdots + v_{m-2,m-1})$$

be two characters of $V_{1^{m-1},r-m+1}$. Notice that $(V_{1^{m-1},r-m+1}, \psi_{m-1})$ is the same as $(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})$ where $\mathcal{O} = (m1^{r-m})$.

We consider slightly more general characters. Let $m' \geq m$, and $\underline{\epsilon} = (\epsilon_m, \epsilon_{m+1}, \cdots, \epsilon_{m'-1}) \in F^{m'-m}$. Let

$$\psi_{m-1,\underline{\epsilon}}(v) = \psi(v_{1,2} + \cdots + v_{m-1,m} + \epsilon_m v_{m,m+1} + \cdots + \epsilon_{m'-1} v_{m'-1,m'})$$

be a character of $V_{1^{m'-1},r-m'+1}$.

Lemma 5.1.6. *If $m > n$, then*

$$J_{V_{1^{m'-1},r-m'+1},\psi_{m-1,\underline{\epsilon}}}(\Theta) = 0.$$

In particular,

$$J_{V_{1^{m-1},r-m+1},\psi_{m-1}}(\Theta) = 0.$$

Proof. We prove this by induction on $r - m'$. When $r = m' \geq m$, the pair $(V_{1^{r-1},1}, \psi_{r-1,\underline{\epsilon}})$ can only be (U, ψ_λ) where λ is a partition of the form $(m'' \cdots)$ with some $m'' \geq m$. The result follows Corollary 3.6.5 since $m'' \geq m > n$.

Now assume the result is true for m' and we prove it $m' - 1$ if $m' - 1 \geq m$. Define $R_{m'-1}$ to be the subgroup of U such that any element $u = (u_{j,l}) \in R_{m'-1}$, $u_{j,l} = 0$, unless $j = m' - 1$. The group $R_{m'-1}$ acts on $V_{1^{m'-2},r-m'+2}$. For any character ξ of $R_{m'-1}$,

$$J_{R_{m'-1},\xi}(J_{V_{1^{m'-2},r-m'+2},\psi_{m'-2,\underline{\epsilon}}}(\Theta)) = 0$$

by induction. This implies

$$J_{V_{1^{m'-2},r-m'+2},\psi_{m'-2,\underline{\epsilon}}}(\Theta) = 0.$$

This finishes the proof. □

Lemma 5.1.7.

$$J_{V_{1^{n-1},r-n+1},\psi_{n-1}}(\Theta) \cong J_{V_{1^n,r-n},\tilde{\psi}_n}(\Theta).$$

Proof. The group R_n acts on $V_{1^{n-1},r-n+1}$. For any nontrivial character ξ of R_n,

$$J_{R_n,\xi}(J_{V_{1^{n-1},r-n+1},\psi_{n-1}}(\Theta)) = 0$$

by Lemma 5.1.6. Therefore, the action of R_n on $J_{V_{1^{n-1},r-n+1},\psi_{n-1}}(\Theta)$ is trivial, and

$$J_{V_{1^{n-1},r-n+1},\psi_{n-1}}(\Theta) \cong J_{V_{1^n,r-n},\tilde{\psi}_n}(\Theta).$$

□

Now we are ready to prove Theorem 5.1.1 part (1). Indeed, since $p_1 > n$,

$$J_{U_\mathcal{O},\psi_\mathcal{O}}(\Theta) = J_*(J_{V_{1^{p_1-1},r-p_1+1},\psi_{p_1-1}}(\Theta)) = 0.$$

Here $*$ is some unipotent subgroup of $V_2(\mathcal{O})$. By Lemma 5.1.4, this implies

$$J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta) = 0.$$

5.1.4 Nonvanishing results

In this subsection, $\mathcal{O} = (n^a b)$. For $1 \le i \le a$, consider $V_{1^{in-1},r-in+1}$ and its characters attached to the partitions (n^i) and $(n^{i-1}(n+1))$, respectively. We can prove the following lemma by using the same arguments in Lemma 5.1.6 and 5.1.7.

Lemma 5.1.8.

(1) $J_{V_{1^{in},r-in},\psi_{(n^{i-1}(n+1))}}(\Theta) = 0.$

(2) $J_{V_{1^{in-1},r-in+1},\psi_{(n^i)}}(\Theta) \cong J_{V_{1^{in},r-in},\psi_{(n^i)}}(\Theta).$

Now we prove the following nonvanishing result (Theorem 5.1.1 part (2)).

Proposition 5.1.9. $J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta) \neq 0.$

Proof. It suffices to show that $J_{U_{\mathcal{O}},\psi_{U_{\mathcal{O}}}}(\Theta) \neq 0$. Indeed,

$$\begin{aligned} J_{U_{\mathcal{O}},\psi_{U_{\mathcal{O}}}}(\Theta) =& J_{U_2((n^{a-1}b)),\psi_{U_2((n^{a-1}b))}}(J_{V_{1^{n-1},r-n+1},\psi_{(n)}}(\Theta)) \\ \cong& J_{U_2((n^{a-1}b)),\psi_{U_2((n^{a-1}b))}}(J_{V_{1^n,r-n},\psi_{(n)}}(\Theta)). \end{aligned}$$

Here, $U_2((n^{a-1}b))$ is viewed as a subgroup of U via the embedding $u \mapsto \mathrm{diag}(I_n, u)$. Now we apply root exchange in $U_2((n^{a-1}b))$. The root exchange does not change anything we did in the previous step. Thus,

$$J_{U_2((n^{a-1}b)),\psi_{U_2((n^{a-1}b))}}(J_{V_{1^n,r-n},\psi_{(n)}}(\Theta)) \neq 0$$

if and only if

$$J_{U_{(n^{a-1}b)},\psi_{U_{(n^{a-1}b)}}}(J_{V_{1^n,r-n},\psi_{(n)}}(\Theta)) \neq 0.$$

Here, $U_{(n^{a-1}b)}$ is again viewed as a subgroup of U via the same embedding. By Lemma 5.1.8,

$$\begin{aligned} & J_{U_{(n^{a-1}b)},\psi_{U_{(n^{a-1}b)}}}(J_{V_{1^n,r-n},\psi_{(n)}}(\Theta)) \\ =& J_{U_2((n^{a-2}b)),\psi_{U_2((n^{a-2}b))}}(J_{V_{1^{2n-1},r-2n+1},\psi_{(n^2)}}(\Theta)) \\ \cong& J_{U_2((n^{a-2}b)),\psi_{U_2((n^{a-2}b))}}(J_{V_{1^{2n},r-2n},\psi_{(n^2)}}(\Theta)). \end{aligned}$$

Here, $U_2((n^{a-2}b))$ is viewed as a subgroup of U via $u \mapsto \mathrm{diag}(I_{2n}, u)$.

Now we repeat this process inductively. This implies that $J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta) \neq 0$ if and only if

$$0 \neq J_{U_2((b)),\psi_{U_2((b))}}(J_{V_{1^{an},r-an},\psi_{(n^a)}}(\Theta)) = J_{U,\psi_{((n^a b))}}(\Theta).$$

The result follows from the nonvanishing results of the semi-Whittaker functionals (Corollary 3.6.7). □

Now suppose that n, b have the same parity. By Lemma 5.1.5 part (3), in all the steps of the above proof, we actually obtain isomorphisms of twisted Jacquet modules, instead of "if and only if" statements. This proves the following result.

Proposition 5.1.10. *When n and b have the same parity,*

$$J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta) \cong J_{U,\psi_\lambda}(\Theta),$$

where λ is the partition $(n^a b)$.

When r is a multiple of n, combining with Corollary 3.6.9, we obtain the following uniqueness result.

Theorem 5.1.11. *When $r = mn$ and $\mathcal{O} = (n^m)$,*

$$\dim J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta) = 1.$$

Remark 5.1.12. We already proved the local results at good primes. At bad primes, these statements would be valid once we have the corresponding vanishing results of semi-Whittaker functionals.

This new unique functional in Theorem 3.6.9 is valuable and it already finds applications in Rankin-Selberg integrals for covering groups. The first – doubling constructions for covering groups – will be discussed in Section 6.1. This unique functional also plays a role in a new-way integral (Euler products with non-unique models) for covering groups; see Ginzburg [Gin16].

5.2 Unipotent Orbits: Global Results

We are back to the global situation in this section. Let $\Theta = \Theta_r$ be the global theta representation on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$, defined in Section 4.1. Let $(n^a b)$ be the unipotent orbit of GL_r as in Section 5.1.

Theorem 5.2.1. *The unipotent orbit attached to Θ is $(n^a b)$. In other words, $\mathcal{O}(\Theta) = (n^a b)$.*

This follows from Proposition 5.2.3 and 5.2.4.

5.2.1 Root exchange lemma: global version

The following global root exchange lemma is proved in [JL13] Lemma 5.2; see also Ginzburg-Rallis-Soudry [GRS11] Section 7.1. This is the global version of Lemma 5.1.3.

Let C be an F-subgroup of a maximal unipotent subgroup of GL_r, and let ψ_C be a nontrivial character of $C(F)\backslash C(\mathbb{A})$. Let $\tilde{X}, \tilde{Y}$ be two unipotent F-subgroups, satisfying the following conditions:

(a) $\tilde{X}$ and $\tilde{Y}$ normalize C;

(b) $\tilde{X}\cap C$ and $\tilde{Y}\cap C$ are normal in $\tilde{X}$ and $\tilde{Y}$, respectively; $(\tilde{X}\cap C)\backslash\tilde{X}$ and $(\tilde{Y}\cap C)\backslash\tilde{Y}$ are abelian;

(c) $\tilde{X}(\mathbb{A})$ and $\tilde{Y}(\mathbb{A})$ preserve ψ_C;

(d) ψ_C is trivial on $(\tilde{X}\cap C)(\mathbb{A})$ and $(\tilde{Y}\cap C)(\mathbb{A})$;

(e) $[\tilde{X},\tilde{Y}]\subset C$;

(f) there is a nondegenerate pairing $(\tilde{X}\cap C)(\mathbb{A})\times(\tilde{Y}\cap C)(\mathbb{A})\to\mathbb{C}^\times$, given by $(x,y)\mapsto\psi_C([x,y])$, which is multiplicative in each coordinate, and identifies $(\tilde{Y}\cap C)(F)\backslash\tilde{Y}(F)$ with the dual of $\tilde{X}(F)(\tilde{X}\cap C)(\mathbb{A})\backslash\tilde{X}(\mathbb{A})$, and $(\tilde{X}\cap C)(F)\backslash\tilde{X}(F)$ with the dual of $\tilde{Y}(F)(\tilde{Y}\cap C)(\mathbb{A})\backslash\tilde{Y}(\mathbb{A})$.

Let $B = C\tilde{Y}$ and $D = C\tilde{X}$, and extend ψ_C trivially to characters of $B(F)\backslash B(\mathbb{A})$ and $D(F)\backslash D(\mathbb{A})$, which are denoted by ψ_B and ψ_D, respectively.

Lemma 5.2.2. *Assume the quadruple $(C,\psi_C,\tilde{X},\tilde{Y})$ satisfies the above conditions. Let f be an automorphic form on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. Then*

$$\int_{C(F)\backslash C(\mathbb{A})} f(cg)\psi_C^{-1}(c)\,dc\equiv 0,\qquad \forall g\in\widetilde{\mathrm{GL}}_r(\mathbb{A}),$$

if and only if

$$\int_{B(F)\backslash B(\mathbb{A})} f(ug)\psi_C^{-1}(u)\,du\equiv 0,\qquad \forall g\in\widetilde{\mathrm{GL}}_r(\mathbb{A}),$$

if and only if

$$\int_{D(F)\backslash D(\mathbb{A})} f(ug)\psi_C^{-1}(u)\,du\equiv 0,\qquad \forall g\in\widetilde{\mathrm{GL}}_r(\mathbb{A}).$$

5.2.2 Vanishing results

Proposition 5.2.3. *Let θ be in the space of Θ_r. Let $\mathcal{O}$ be a unipotent orbit which is greater than or not comparable to $(n^a b)$. Then the integral*

$$\int_{U_2(\mathcal{O})(F)\backslash U_2(\mathcal{O})(\mathbb{A})} \theta(ug)\psi_{U_2(\mathcal{O})}(u)\, du$$

is zero for all choices of data.

Proof. As in the case of semi-Whittaker coefficients, the global vanishing result follows from the local vanishing result. Let v be a nonarchimedean place of F such that $|n|_v = 1$ and Θ_r is unramified at v. If

$$\int_{U_2(\mathcal{O})(F)\backslash U_2(\mathcal{O})(\mathbb{A})} \theta(ug)\psi_{U_2(\mathcal{O})}(u)\, du$$

is nonzero, then the functional

$$\theta \longmapsto \int_{U_2(\mathcal{O})(F)\backslash U_2(\mathcal{O})(\mathbb{A})} \theta(ug)\psi_{U_2(\mathcal{O})}(u)\, du$$

induces a nonzero functional on $\Theta_{r,v}$ which factors through the twisted Jacquet module for the character $\psi_{U_2(\mathcal{O})}(u)$ on the group $U_2(\mathcal{O})(F_v)$. This contradicts the local result. $\square$

5.2.3 Nonvanishing results

Proposition 5.2.4. *Let θ be in the space of Θ_r. Let $\mathcal{O} = (n^a b)$. Then the integral*

$$\int_{U_2(\mathcal{O})(F)\backslash U_2(\mathcal{O})(\mathbb{A})} \theta(ug)\psi_{U_2(\mathcal{O})}(u)\, du$$

is nonzero for some choice of data.

Proof. The proof is analogous to the local case. Once we have the global root exchange lemma and global vanishing results, the nonvanishing results follow from the corresponding nonvanishing results on the semi-Whittaker coefficients. Notice that the global version of Lemma 5.1.6, 5.1.7 and 5.1.8 can be established by using the corresponding local results. We omit the details. We remark that a more general result on relations between these two types of Fourier coefficients can be found in [Cai16a, GGS16]. $\square$

Chapter 6

Doubling Constructions for Covering Groups

In the 1980's, Piatetski-Shapiro and Rallis [GPSR87] discovered a family of Rankin-Selberg integrals for the classical groups that did not rely on Whittaker models. This is the so-called doubling method. In the joint work with Friedberg, Ginzburg and Kaplan [CFGK17], we give a generalization of the doubling method. We present a family of integrals representing tensor product L-functions of classical groups with general linear groups. Our construction is uniform over all classical groups and their non-linear coverings, and is applicable to all cuspidal representations.

In this chapter, we present some details for the first covering case. That is the standard L-function for an irreducible genuine cuspidal automorphic representation on the 3-fold cover of Sp_2.

6.1 Whittaker-Speh-Shalika representations

The global integral given in [CFGK16, CFGK17] relies on the following unique models: matrix coefficients and a degenerate type unique model on general linear groups. We recall the definition here.

Definition 6.1.1. An irreducible genuine automorphic representation π of $\widetilde{\mathrm{GL}}_{ab}(\mathbb{A})$ is a Whittaker-Speh-Shalika representation of type (a, b) if:

(1) $\mathcal{O}(\pi) = (a^b)$.

(2) For a finite place v, let π_v denote the irreducible constituent of π at v. Suppose that π_v is an unramified representation. Then $\mathcal{O}(\pi_v) = (a^b)$. (That is, the local analogue of part (1) holds.) Moreover,

$$\dim \mathrm{Hom}_{U_2((a^b))(F_v)}(\pi_v, \psi_{U_2((a^b))}) = 1. \tag{6.1}$$

Thus, when r is a multiple of n, we can rephrase Theorems 5.1.2, 5.1.11 and 5.2.1 as follows.

Theorem 6.1.2. *When $r = mn$, the representation Θ is a Whittaker-Speh-Shalika representation of type (n, m).*

6.2 Global integral

In this section, we give a global integral that represents the standard L-function for cubic cover of Sp_2. All the covers will be cubic covers in this section.

Let π be an irreducible genuine cuspidal automorphic representation on $\widetilde{Sp}_2(\mathbb{A})$. Recall that the dual group of $\widetilde{Sp}_2(\mathbb{A})$ is $Sp_2(\mathbb{C})$. Thus, we can define the standard partial L-function $L^S(s, \pi)$ as an Euler product over places outside a finite set of places S.

Recall that the unipotent orbit attached to the theta representation Θ_6 is (3^2) and at unramified places, such a model is unique. Consider the Siegel parabolic of Sp_{12}, and use Θ_6 to construct an Eisenstein series $E(g, s)$ on $\widetilde{Sp}_{12}(\mathbb{A})$. We consider the following integral:

$$\int\limits_{Sp_2(F)\times Sp_2(F)\backslash Sp_2(\mathbb{A})\times Sp_2(\mathbb{A})} \int\limits_{U_0(F)\backslash U_0(\mathbb{A})} \varphi_1(g_1)\overline{\varphi_2(g_2)}E(u\iota(g_1, g_2), s)\psi_{U_0}(u)\ du\ dg_1\ dg_2.$$

Here, $\varphi_1, \varphi_2 \in \pi$. The unipotent group U_0 and character ψ_{U_0} are defined as follows. Given a matrix, let $X[i, j]$ denote its (i, j)-coordinate. The group U_0 is the unipotent radical of the standard parabolic subgroup whose Levi subgroup is $\mathrm{GL}_2 \times \mathrm{GL}_2 \times Sp_4$.

In terms of matrices, we write

$$U_0 = \left\{ u = \begin{pmatrix} I_2 & X_1 & * & * & * \\ & I_2 & X_2 & * & * \\ & & I_4 & * & * \\ & & & I_2 & * \\ & & & & I_2 \end{pmatrix} \in Sp_{12} \right\}$$

and

$$\psi_{U_0}(u) = \psi(\mathrm{tr}(X_1) + X_2[1,1] + X_2[2,4]).$$

The "doubling" map

$$\iota : Sp_2 \times Sp_2 \to Sp_{12}$$

is given by

$$(g_1, g_2) \mapsto \begin{pmatrix} g_1 & & & & & & \\ & g_1 & & & & & \\ & & g_{11} & & g_{12} & & \\ & & & g_2 & & & \\ & & g_{13} & & g_{14} & & \\ & & & & & g_1^* & \\ & & & & & & g_1^* \end{pmatrix},$$

where $g_1 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$.

When $\mathrm{Re}(s) \gg 0$, this integral is absolutely convergent. We plug in the definition of the Eisenstein series. After unfolding, we can show that the above integral is

$$\int\limits_{Sp_2(\mathbb{A})} \int\limits_{U_1(\mathbb{A})} \langle \varphi_1, \pi(g)\varphi_2 \rangle f_{W(\Theta)}(\delta u_0 \iota(1,g), s) \psi_{U_0}(u_1) \, du_1 \, dg.$$

Here the matrix coefficient is given by

$$\langle \varphi_1, \pi(g)\varphi_2 \rangle = \int\limits_{Sp_2(F)\backslash Sp_2(\mathbb{A})} \varphi_1(g_1)\overline{\varphi_2(g_1 g)} \, dg_1,$$

the element δ is

$$\begin{pmatrix} & I_6 \\ -I_6 & \end{pmatrix} \begin{pmatrix} I_4 & & & \\ & I_2 & I_2 & \\ & & I_2 & \\ & & & I_4 \end{pmatrix},$$

and U_1 is the subgroup of U_0 consisting of elements of the form

$$\begin{pmatrix} I_2 & & & * & * & * \\ & I_2 & & * & * & * \\ & & I_2 & & * & * \\ & & & I_2 & & \\ & & & & I_2 & \\ & & & & & I_2 \end{pmatrix}.$$

The global integral is Eulerian, thanks to the unique model on Θ_6. At unramified places, the corresponding local integral represents

$$\frac{L(21s-10,\pi)}{\zeta(21s-7)\zeta(42s-20)\zeta(42s-18)\zeta(42s-16)},$$

where $\zeta(\cdot)$ is the Dedekind zeta function for F.

Bibliography

[BFG03] Daniel Bump, Solomon Friedberg, and David Ginzburg. Small representations for odd orthogonal groups. *Int. Math. Res. Not.*, (25):1363–1393, 2003.

[BG92] Daniel Bump and David Ginzburg. Symmetric square L-functions on $\mathrm{GL}(r)$. *Ann. of Math. (2)*, 136(1):137–205, 1992.

[BGH96] Daniel Bump, David Ginzburg, and Jeffrey Hoffstein. The symmetric cube. *Invent. Math.*, 125(3):413–449, 1996.

[BLS99] William D. Banks, Jason Levy, and Mark R. Sepanski. Block-compatible metaplectic cocycles. *J. Reine Angew. Math.*, 507:131–163, 1999.

[BZ77] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive $\mathfrak{p}$-adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977.

[Cai16a] Yuanqing Cai. Fourier coefficients for degenerate Eisenstein series and the descending decomposition. *arXiv:1606.08872*, 2016.

[Cai16b] Yuanqing Cai. Fourier coefficients for Theta representations on covers of general linear groups. *arXiv:1602.06614*, 2016.

[Car93] Roger W. Carter. *Finite groups of Lie type*. Wiley Classics Library. John Wiley & Sons, Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[CFGK16] Yuanqing Cai, Solomon Friedberg, David Ginzburg, and Eyal Kaplan. Doubling constructions for covering groups and tensor product L-functions. *arXiv:1601.08240*, 2016.

[CFGK17] Yuanqing Cai, Solomon Friedberg, David Ginzburg, and Eyal Kaplan. Doubling constructions for covering groups and tensor product L-functions. *preprint*, 2017.

[CM93] David H. Collingwood and William M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[CO13] Gautam Chinta and Omer Offen. A metaplectic Casselman-Shalika formula for GL_r. *Amer. J. Math.*, 135(2):403–441, 2013.

[FGar] Solomon Friedberg and David Ginzburg. Descent and theta functions for metaplectic groups. *Journal of the European Mathematical Society*, to appear.

[FKS90] Y. Flicker, D. Kazhdan, and G. Savin. Explicit realization of a metaplectic representation. *J. Analyse Math.*, 55:17–39, 1990.

[Gan16] Wee Teck Gan. The metaplectic tensor product as an instance of Langlands functoriality. *preprint*, 2016.

[GGS16] Raul Gomez, Dmitry Gourevitch, and Siddhartha Sahi. Whittaker supports for representations of reductive groups. *arXiv preprint arXiv:1610.00284*, 2016.

[Gin06] David Ginzburg. Certain conjectures relating unipotent orbits to automorphic representations. *Israel J. Math.*, 151:323–355, 2006.

[Gin14] David Ginzburg. Towards a classification of global integral constructions and functorial liftings using the small representations method. *Adv. Math.*, 254:157–186, 2014.

[Gin15] David Ginzburg. On certain global constructions of automorphic forms related to small representations of F_4. *arXiv:1503.06409*, 2015.

[Gin16] David Ginzburg. Generating functions on covering groups. *arXiv:1603.05784*, 2016.

[GJ78] Stephen Gelbart and Hervé Jacquet. A relation between automorphic representations of GL(2) and GL(3). *Ann. Sci. École Norm. Sup. (4)*, 11(4):471–542, 1978.

[GK82] S. S. Gelbart and A. W. Knapp. L-indistinguishability and R groups for the special linear group. *Adv. in Math.*, 43(2):101–121, 1982.

[GPSR87] Stephen Gelbart, Ilya Piatetski-Shapiro, and Stephen Rallis. *Explicit constructions of automorphic L-functions*, volume 1254 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[GRS99] David Ginzburg, Stephen Rallis, and David Soudry. On a correspondence between cuspidal representations of GL_{2n} and $\widetilde{\mathrm{Sp}}_{2n}$. *J. Amer. Math. Soc.*, 12(3):849–907, 1999.

[GRS11] David Ginzburg, Stephen Rallis, and David Soudry. *The descent map from automorphic representations of* $\mathrm{GL}(n)$ *to classical groups*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[JL13] Dihua Jiang and Baiying Liu. On Fourier coefficients of automorphic forms of $\mathrm{GL}(n)$. *Int. Math. Res. Not. IMRN*, (17):4029–4071, 2013.

[Kab01] Anthony C. Kable. The tensor product of exceptional representations on the general linear group. *Ann. Sci. École Norm. Sup. (4)*, 34(5):741–769, 2001.

[Kapar] Eyal Kaplan. The characterization of theta-distinguished representations of GL_n. *Israel J. Math*, to appear.

[KP84] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.

[McN12] Peter J. McNamara. Principal series representations of metaplectic groups over local fields. In *Multiple Dirichlet series, L-functions and automorphic forms*, volume 300 of *Progr. Math.*, pages 299–327. Birkhäuser/Springer, New York, 2012.

[Mez04] Paul Mezo. Metaplectic tensor products for irreducible representations. *Pacific J. Math.*, 215(1):85–96, 2004.

[Mœg96] C. Mœglin. Front d'onde des représentations des groupes classiques p-adiques. *Amer. J. Math.*, 118(6):1313–1346, 1996.

[MW87] C. Mœglin and J.-L. Waldspurger. Modèles de Whittaker dégénérés pour des groupes p-adiques. *Math. Z.*, 196(3):427–452, 1987.

[MW95] C. Mœglin and J.-L. Waldspurger. *Spectral decomposition and Eisenstein series*, volume 113 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1995. Une paraphrase de l'Écriture [A paraphrase of Scripture].

[OS08] Omer Offen and Eitan Sayag. Global mixed periods and local Klyachko models for the general linear group. *Int. Math. Res. Not. IMRN*, (1):Art. ID rnm 136, 25, 2008.

[PPS89] S. J. Patterson and I. I. Piatetski-Shapiro. The symmetric-square L-function attached to a cuspidal automorphic representation of GL_3. *Math. Ann.*, 283(4):551–572, 1989.

[Shi75] Goro Shimura. On the holomorphy of certain Dirichlet series. *Proc. London Math. Soc. (3)*, 31(1):79–98, 1975.

[Tak14] Shuichiro Takeda. The twisted symmetric square L-function of GL(r). *Duke Math. J.*, 163(1):175–266, 2014.

[Tak16] Shuichiro Takeda. Metaplectic tensor products for automorphic representation of $\widetilde{GL}(r)$. *Canad. J. Math.*, 68(1):179–240, 2016.

[Takar] Shuichiro Takeda. Remarks on metaplectic tensor products for covers of $\mathrm{GL}(r)$. *Pacific J. Math.*, to appear.